DISTORTION BOUNDS FOR $C^{2+\eta}$ UNIMODAL MAPS

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Abstract. We obtain estimates for derivative and cross–ratio distortion for $C^{2+\eta}$ (any $\eta > 0$) unimodal maps with non–flat critical points. We do not require any 'Schwarzian–like' condition.

For two intervals $J \subset T$, the cross–ratio is defined as the value

$$B(T, J) := \frac{|T||J|}{|L||R|}$$

where L, R are the left and right connected components of $T \setminus J$ respectively. For an interval map g such that $g_T : T \to \mathbb{R}$ is a diffeomorphism, we consider the cross–ratio distortion to be

$$B(g, T, J) := \frac{B(g(T), g(J))}{B(T, J)}.$$

We prove that for all $0 < K < 1$ there exists some interval I_0 around the critical point such that for any intervals $J \subset T$, if $f^n|_T$ is a diffeomorphism and $f^n(T) \subset I_0$ then

$$B(f^n, T, J) > K.$$

Then the distortion of derivatives of $f^n|_J$ can be estimated with the Koebe Lemma in terms of K and $B(f^n(T), f^n(J))$. This tool is commonly used to study topological, geometric and ergodic properties of f. This extends a result of Kozlovski.

1. Introduction

In order to understand the long term behaviour of smooth dynamical system $f : X \to X$ we must consider iterates of the map. It is useful to know how differently high iterates of the map f^n act on nearby points. For example we can try to estimate how wild the derivative of iterates of the map is: we can consider the distortion $\frac{Df^n(x)}{Df^n(y)}$ for x, y in some small interval J where $f^n|_J$ is a diffeomorphism. For one dimensional maps, the Koebe Lemma is a tool we use to estimate this. Notice that this distortion can be rather wild when f has critical points.

An important condition we must assume in order to apply the Koebe Lemma is that the map f^n must increase cross–ratios. The type of cross–ratio we use most is defined as follows. For two intervals $J \subset T$, the *cross–ratio* is

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defined as the value

$$B(T,J) := \frac{|T||J|}{|L||R|}$$

where L, R are the left and right connected components of $T \setminus J$ respectively. For an interval map g such that $g_T : T \to \mathbb{R}$ is a diffeomorphism, the main measure of cross–ratio distortion we use is given by

$$B(g,T,J) := \frac{B(g(T), g(J))}{B(T,J)}.$$

If we know that $B(f^n, T^*, J^*) \geqslant K > 0$ for any $J^* \subset T^* \subset T$ then we have uniform bounds on $\frac{Df^n(x)}{Df^n(y)}$ for $x, y \in J$ depending on K and $B(f^n(T), f^n(J))$. So we are able to estimate the distortion of the derivative of f^n using information on the distortion of the cross–ratios.

A classical way of gaining information about the dynamics of an interval map $f : [0,1] \to [0,1]$ with a critical point, is to take a first return map to some well chosen interval I. If this map has some diffeomorphic branches, we can estimate how well or how badly the derivatives behave on branches using the Koebe Lemma as above. This method is often used to give information on the geometry and topology of the map and its iterates, see [**MS**]. This type of approach is also applied when considering the ergodic properties of one dimensional maps. Often instead of first return maps, certain *inducing schemes* are applied in these cases, see [**MS**]. The Koebe Lemma allows us to show that the inducing schemes are expansive, and the Folklore Theorem can then be used to derive ergodic absolutely continuous f–invariant measures.

In order to apply the Koebe Lemma to $f^n|_T$ we need a lower bound on cross ratio distortion of $f^n|_T$. In fact, a lower bound $K = 1$ is obtained whenever f is C^3 and has negative *Schwarzian derivative*: that is

$$Sf := \frac{D^3 f}{Df} - \frac{3}{2}\left(\frac{D^2 f}{Df}\right)^2$$

is negative wherever it is well defined. For applications it is not so important that f have negative Schwarzian, just that some iterate of f has negative Schwarzian on some small intervals. Kozlovski showed [**K2**] that for any C^3 unimodal map with non–flat critical point (see the next section), if I is a small enough neighbourhood of the critical point and $f^n(x) \in I$ then $Sf^{n+1}(x) < 0$. Therefore, for most practical purposes, for example where first return maps or inducing schemes are used to gain information about the dynamics, it is unnecessary to find the sign of the Schwarzian derivative as long as the critical point is non–flat. Moreover, this result allowed Kozlovski to prove the following.

Theorem 1.1. *Suppose that f is a C^3 unimodal map with non–flat critical point whose iterates do not converge to a periodic attractor. Then for any*

$0 < K < 1$, there is an interval V around the critical point such that if, for an interval T and some $n > 0$,

- *$f^n|_T$ is monotone; and*
- *each interval from the orbit $\{T, f(T), \ldots, f^n(T)\}$ is contained in the domain of the first entry map to V,*

then

$$B(f^n, T, J) > K$$

where J is any subinterval of T.

This means that the Koebe Lemma can be applied to f^n to get estimates on the distortion of derivatives which only depend on $B(f^n(T), f^n(J))$ (for first return maps or induced maps this quantity is bounded whenever the branches have a 'uniform extension'). These results were extended to C^3 multimodal maps with non–flat critical points in [**SV**]. Also, for C^3 unimodal maps with non–flat critical point, it is shown in [**GSS2**] that an analytic coordinate change can create a map which has first return maps with negative Schwarzian.

So how necessary is the negative Schwarzian condition to prove dynamical results in 'reasonable' cases? Certainly it is useful in determining the type of parabolic periodic points or bounding the number of attracting cycles, see [**Si**, **MS**]. A natural question to ask, and the one we consider in this paper, is: what happens for unimodal maps with non–flat critical points which are not C^3? Certainly the usual negative Schwarzian condition is no use since it is not even defined. (Note that there is a 'Schwarzian–like' condition for C^1 maps -equivalent to the negative Schwarzian condition when the map is C^3- see [**P**, **MS**], but that need not hold in our case either.) We show that Theorem 1.1 extends to the case of $C^{2+\eta}$ for any $\eta > 0$. So many results on the geometric and statistical properties of unimodal maps with non–flat critical point extend to maps which are only $C^{2+\eta}$.

Since we cannot use the negative Schwarzian property at all here, we must look rather closely at the behaviour of the map on small scales. We use a result in [**MS**] to estimate the cross ratio distortion in terms of sums of lengths of intervals. We split up this sum into blocks using the domains of first return maps to small intervals around the critical point. The precise behaviour of the branch containing the critical point, the central branch, determines how we choose our blocks. Since we have no negative Schwarzian property, there are particular difficulties when a block of our sum contains points which spend a very long time in the central branch (when there is a so called 'saddle node cascade' or an 'Ulam-Neumann cascade'). The main tool we use here is the *real bounds* proved by [**V**, **Sh1**, **SV**]. Roughly speaking, these results give us a sequence of first return maps where the diffeomorphic branches have a uniformly large extension. This gives bounded distortion

of the derivative on these branches which allows us to estimate the sums of lengths of intervals.

1.1. **Statement of the main result.** We explain the terminology in the following definitions. Given an interval T, and a subinterval $J \subset T$, we defined the cross–ratio $B(T, J)$ above. Note that if we again denote the left–hand and right–hand components of $T \setminus J$ by L and R respectively, we have another measure of cross–ratio
$$A(T, J) := \frac{|T||J|}{|L \cup J||J \cup R|},$$
(however, we focus mainly on $B(T, J)$).

Suppose that $g : T \to \mathbb{R}$ is a diffeomorphism. We define $B(g, T, J)$ as above, but we also have
$$A(g, T, J) := \frac{A(g(T), g(J))}{A(T, J)},$$
another estimate of how the map distorts cross–ratios. Observe that for diffeomorphisms $g : T \to g(T)$ and $h : g(T) \to h \circ g(T)$ we have
$$B(h \circ g, T, J) = B(h, g(T), g(J))B(g, T, J).$$
Similarly for $A(g, T, J)$.

We say that T is a *δ–scaled neighbourhood of* J if $\frac{|L|}{|J|}, \frac{|R|}{|J|} > \delta$. We suppose throughout that our functions map from $I := [0, 1]$ into itself, and ∂I into ∂I.

We say that a unimodal C^k map g has *non–flat* critical point c if there exists some neighbourhood U of c and a C^k diffeomorphism $\phi : U \to I$ with $\phi(c) = 0$ such that $g(x) = \pm|\phi(x)|^\alpha + g(c)$ for some $\alpha > 1$. The value α is known as the *critical order* for g. We denote the set of such maps by NF^k and this neighbourhood by U_ϕ.

Such maps have many good properties. For example, they have no wandering intervals, see for example Chapter IV of [**MS**]. More importantly for us here is how such maps distort cross–ratios. In particular, how iterates of such maps distort cross–ratios. Our main result is as follows.

Theorem 1.2. *For any $\eta > 0$, let $f \in NF^{2+\eta}$ be a unimodal map with a critical point whose iterates do not converge to a periodic attractor. Then for any $0 < K < 1$, there is an interval V around the critical point such that if, for an interval T and some $n > 0$,*

- *$f^n|_T$ is monotone; and*
- *$f^n(T) \subset V$,*

then
$$B(f^n, T, J) > K,$$

$$A(f^n, T, J) > K$$

where J is any subinterval of T.

This theorem is proved for C^3 maps in [**K2**]. Note that in fact we prove that if $0 < \eta \leqslant 1$ then for any $0 < \eta' < \eta$, there exists $C > 0$ such that if J, T, V are as in the theorem then $A(f^n, T, J), B(f^n, T, J) > \exp\{-C(\sup_j |V_j|)^{\eta'}\}$.

1.2. **Strategy of the proof.** Our setup will involve first return maps to a neighbourhood of c, as outlined below. For the case where c is non–recurrent see [**St**]. So we suppose throughout that c is recurrent.

An open interval V is *nice for f* if $f^n(\partial V) \cap V = \emptyset$ for $n \geqslant 1$. (When it is clear what f is, we just refer to such interval as nice.) It is easy to see that we can find arbitrarily small nice intervals around c.

Let $I_0 \ni c$ be a nice interval. For every $x \in I$ whose orbit intersects I_0, let $n(x) := \min\{k > 0 : f^k(x) \in I_0\}$. If additionally $x \in I_0$, let $I_0^j \ni x$ be the maximal neighbourhood such that $f^{n(x)}(I_0^j) \subset I_0$. We obtain the first return map $F_0 : \bigcup_j I_0^j \to I_0$. We label the interval which contains c by I_0^0; this interval is called the *central domain*. Observe that F_0 is a diffeomorphism on all domains I_0^j except when $j = 0$. F_0 is unimodal on I_0^0. Note also that I_0^0 is again a nice interval. We will call it I_1 for the next step in the inducing process; i.e. we define $F_1 : \bigcup_j I_1^j \to I_1$ to be the first return map to $I_1 = I_0^0$. It has central domain $I_1^0 = I_2$. Continuing inductively, we obtain maps $F_i : \bigcup_j I_i^j \to I_i$. The sequence $I_0 \supset I_1 \supset \cdots$ is called the *principal nest*, and $F_i|_{I_i^j} : I_i^j \to I_i$ is a *branch* of F_i.

If $x \notin I_i$ but $n(x)$ is defined then there is a maximal interval $U_i^j \ni x$ such that $f^{n(x)} : U_i^j \to I_i$ is a diffeomorphism. So we may extend F_i, letting $F_i|_{U_i^j} : U_i^j \to I_i$. Then letting $\bigcup_j U_i^j$ consist of all such intervals added to $\bigcup_j I_i^j$, we call $F_i : \bigcup_j U_i^j \to I_i$ the *first entry map to I_i*. We will often switch between these two very similar types of map.

For simplicity, except in the appendix, we will assume that $F_i(c)$ is a maximum for $F_i|_{I_{i+1}}$. We say that F_i is *low* if $F_i(c)$ lies to the left of c and F_i is *high* if $F_i(c)$ lies to the right of c. F_i is *central* if $F_i(c)$ is inside I_{i+1} (if this is not the case, then F_i is *non–central*). Figure 1 shows F_i which is high and central return.

Suppose that $f^n : T \to f^n(T)$ is a diffeomorphism and $f^n(T) \subset I_0$. It can be shown (see Lemma 2.1) that we get a lower bound on $B(f^n, T, J)$ if we can find some bound on $\sum_{k=0}^{n-1} |f^k(T)|$. In fact, we consider $\sum_{k=0}^{n-1} |f^k(T)|^{1+\xi}$ for some $0 < \xi < \eta$. We will split up this sum into blocks determined by the principal nest explained above. Note that our proofs extend easily to $A(f^n, T, J)$, see [**St**].

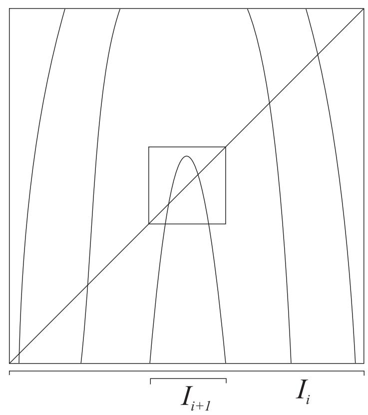

FIGURE 1. F_i is high and central.

We fix n and T as in Theorem 1.2, let $n_0 = n$. For $i > 0$, suppose that some iterate $f^j(T)$ enters I_i for $0 \leqslant j \leqslant n$. Now we let n_i be the last time that $f^j(T) \subset I_i$, i.e. $f^{n_i}(T) \subset I_i$ and $f^{n_i+j}(T) \not\subseteq I_i$, $0 < j \leqslant n - n_i$. If $f^j(T)$ is never contained in I_i for $0 \leqslant j \leqslant n$ then we let $n_i = n_{i-1}$. For each i, we will be interested in estimating

$$\sum_{k=1}^{n_i - n_{i+1}} |f^{k+n_{i+1}}(T)|^{1+\xi} \qquad \text{we call this the \textit{the sum for} } F_i.$$

As we will see later, if F_i is non–central infinitely often then Theorem 2.3 implies that as $i \to \infty$ the intervals I_i shrink down to c. Thus we are able to bound $\sum_{k=0}^{n-1} |f^k(T)|^{1+\xi}$ by bounding the sums for all F_i. We will use a slightly different method when there exists a nice I_0 such that F_i is always central.

In order to prove the main theorem, we will consider the following cases. Note that we only assume that $f \in NF^2$ in the following three propositions.

- F_{i-2} is non–central. We consider the sum for F_i whenever $f^j(T) \cap \partial I_{i+1} = \emptyset$ for all $0 \leqslant j < n_i$, as follows.

  Proposition 1.3. *Suppose that F_{i-2} is non–central and $f^j(T) \cap \partial I_{i+1} = \emptyset$ for all $0 \leqslant j < n_i$. Then there exists $C_{wb} > 0$ such that*

  $$\sum_{k=1}^{n_i - n_{i+1}} |f^{k+n_{i+1}}(T)| < C_{wb} \sigma_i \frac{|f^{n_i}(T)|}{|I_i|},$$

  where $\sigma_i := \sup_{V \in \{I_i^j\}_j} \sum_{k=1}^{n(V)} |f^k(V)|$ (and $n(V)$ is defined as k where $F_i|_V = f^k$).

  We call this a *well bounded case*. It is dealt with in Section 3.

- F_{i-2} is non–central and $F_i, \dots, F_{i+m-1}$ are central. We consider the sums for $F_i, F_{i+1}, \dots, F_{i+m}$ whenever $f^j(T) \cap \partial I_{i+m+1} = \emptyset$ for all $0 \leqslant j < n_i$, as follows.

  Proposition 1.4. *Suppose that F_{i-2} is non–central, $F_i, \dots, F_{i+m-1}$ are central and $f^j(T) \cap \partial I_{i+m+1} = \emptyset$ for all $0 \leqslant j < n_i$. For all $\xi > 0$ there exists $C_{casc} > 0$ such that*
$$\sum_{k=1}^{n_i - n_{i+m+1}} |f^{k+n_{i+m+1}}(T)|^{1+\xi} < C_{casc}\sigma_{i,m} \max_{n_{i+m+1} < k \leqslant n_i} |f^k(T)|^\xi$$
  where $\sigma_{i,m}$ is defined as follows. Let $\sigma_i := \sup_{V \in \{I_i^j\}_j} \sum_{k=1}^{n(V)} |f^k(V)|$. Let $\hat{V} \subset I_i \setminus I_{i+1}$ be an interval such that $f^{\hat{n}}(\hat{V})$ is one of the connected components of $I_i \setminus I_{i+1}$ for some $\hat{n} > 0$ and $f^j(\hat{V})$ is disjoint from both $I_i \setminus I_{i+1}$ and I_m for $0 < j < \hat{n}(\hat{V})$. Then $\sigma_{i,m}$ is the supremum of all such sums $\sum_{j=1}^{\hat{n}(\hat{V})} |f^j(\hat{V})|$ and σ_i.

  We call this the *cascade case*. It is dealt with in Section 4.
- F_{i-2} is central and F_{i-1} is high and non–central. We consider the sum for F_i whenever $f^j(T) \cap \partial I_{i+1} = \emptyset$ for all $0 \leqslant j < n_i$, as follows.

  Proposition 1.5. *Suppose that F_{i-2} is central, F_{i-1} is high and non–central and $f^j(T) \cap \partial I_{i+m+1} = \emptyset$ for all $0 \leqslant j < n_i$. Then there exist $C_{ex} > 0$ and $n_{i+1} < n_{i,3} < n_{i,2} < n_i$ such that $f^{n_{i,2}}(T), f^{n_{i,3}}(T) \subset I_i$ and*
$$\sum_{k=1}^{n_i - n_{i+1}} |f^{k+n_{i+1}}(T)| < C_{ex}\sigma_i \left(\frac{|f^{n_i}(T)|}{|I_i|} + \frac{|f^{n_{i,2}}(T)|}{|I_i|} + \frac{|f^{n_{i,3}}(T)|}{|I_i|} \right).$$

  (In some cases, the last two terms in the sum are not required.) We call this the *exceptional branches case*. It is dealt with in Section 5. We also note there that if F_{i-2} is central and F_{i-1} is low and non–central then we are in another well bounded case, and so the conclusion of Proposition 1.3 holds.
- We have an interval I_0 such that F_i are all central for $i = 0, 1, \dots$. We call this the *infinite cascade case*. We prove Theorem 1.2 for this case in Section 7.

The proof of Theorem 1.2 for the non–infinite cascade case is given in Section 6.

With these propositions, for $0 < \eta' < \eta$, we can decompose the sum $\sum_{k=0}^{n-1} |f^k(T)|^{1+\eta'}$ into blocks of sums $\sum_{k=1}^{n_i - n_{i+1}} |f^{k+n_{i+1}}(T)|^{1+\eta'}$. We then show that each of these is uniformly bounded. We will then show that $\sum_{k=1}^{n_i - n_{i+1}} |f^{k+n_{i+1}}(T)|^{1+\eta}$ decays in a uniform way with i.

The first two cases use real bounds of Theorem 2.3. These bounds imply that $B(I_i^j, I_i)$ are bounded above. This will also be true for all except possibly

two domains of F_i in the third case. The main tool here is Lemma 3.3, which gives us some decay of cross–ratios when we have these real bounds. Note that the conditions $f^j(T)\cap\partial I_{i+1}=\emptyset$ for all $0\leqslant j<n_i$ in well bounded and exceptional cases, and $f^j(T)\cap\partial I_{i+m+1}=\emptyset$ for all $0\leqslant j<n_i$ in the cascade case, make the propositions simpler to prove. However, as we remark in Section 6, it is easy to see how to split up the intervals in the other cases in order to prove Theorem 1.2.

The final case, which arises in the infinitely renormalisable case, is different from the other three. We use a lemma of [**K2**] to find some uniform expanding property which helps bound the sums.

In all cases except the infinite cascade case we must ensure that we have some initial interval which has a first return map which is well bounded. To do this we can simply pick some nice interval to begin with and then induce until we find a map which is well–bounded. This is always possible when there is not an infinite cascade.

Note that we need extra smoothness to bound cross–ratios in the cascade case. This ensures that we can deal with the case when we have many consecutive low central returns, a 'saddle node cascade'.

In Kozlovski's proof for C^3 maps he was able to use the fact that there exists some $C>0$ depending only on f such that for interval $J\subset T$ we have $B(f,T,J)>\exp\{-C|T|^2\}$ and $A(f,T,J)>\exp\{C|L||R|\}$. See Chapter IV.2 of [**MS**]. In particular this means that there exist such real bounds as in Theorem 2.3 for all i, not just those for which F_{i-1} is a non–central return. So the long central cascades we encounter in Section 4 present much less of a problem in the C^3 case. Indeed, the work done in Section 5 is also unnecessary in the C^3 case.

We will deal with the well bounded case first. It is the simplest and gives us a good idea about how we may proceed in general. We will use J to refer to a general interval from here until Section 6. This allows us to use less notation. When we use the constant $C>0$, we mean some constant depending only on f.

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2. Introductory results

Without loss of generality, we suppose throughout that our maps have a maximum at the critical point. We also suppose that f is symmetric about c. That is, $f(c-\epsilon) = f(c+\epsilon)$ for all ϵ. This assumption is useful for simplifying proofs (particularly in Section 5, which is already quite technical), but is not crucial since on small scales our maps will be essentially symmetric (in particular, $|Df(c-\epsilon)|$ and $|Df(c+\epsilon)|$ are arbitrarily close for small enough ϵ). We let $C \leqslant |g|_U \leqslant C'$ mean $\sup_{x\in U} |g(x)| \leqslant C'$ and $\inf_{x\in U} |g(x)| \geqslant C$.

The following theorem is proved for a more general case in Chapter IV of [**MS**]. Here we will let w_g be the modulus of continuity of a continuous map g, i.e. $w_g(\epsilon) := \sup_{|x-y|<\epsilon} |g(x) - g(y)|$.

Theorem 2.1. *For a unimodal map $g : I \to I$, $g \in NF^2$, if T is an interval such that $g^n|_T$ is a diffeomorphism and $J \subset T$ is a subinterval, then there exists some $C > 0$ such that*

$$B(g^n, T, J) > \exp\left\{-C \sum_{i=0}^{n-1} w_{D^2g}(|g^i(T)|)|g^i(T)|\right\}.$$

This bound also holds for $A(f^n, T, J)$.

In Sections 6 and 7 we will use the fact that when $g \in NF^{2+\eta}$ for some $\eta > 0$, we can replace $Cw_{D^2g}(\epsilon)$ by $C\epsilon^\eta$.

The following lemma, a consequence of the absence of wandering intervals, is Lemma 5.2 in [**K2**].

Lemma 2.2. *Suppose that $g \in NF^2$, $g : I \to I$. Then there exists a function $\tau : [0, |I|] \to [0, \infty)$ such that $\lim_{\epsilon\to 0} \tau(\epsilon) = 0$ and for any interval V for which $g^n|_V$ is a diffeomorphism and $g^n(V)$ is disjoint from the immediate basins of periodic attractors, we have*

$$\max_{0\leqslant i\leqslant n} |g^i(V)| < \tau(|g^n(V)|).$$

We may use this lemma and Theorem 2.1 to get

$$B(g^n, T, J) > \exp\left\{-\sigma'(|g^{n-1}(T)|) \sum_{i=0}^{n-1} |g^i(T)|\right\} \tag{1}$$

whenever $f^n(T)$ is disjoint from the immediate basins of periodic attractors, where

$$\sigma'(|g^m(T)|) = Cw_g \circ \tau(|g^m(T)|). \tag{2}$$

We will use the following result of [**SV**] throughout. (In fact it is stated there in greater generality, as Theorem A.)

Theorem 2.3. *If $g \in NF^2$ is a unimodal map with recurrent critical point, then the following hold.*

(a) *For all $k \geqslant 0$ there exists $\xi(k) > 0$ such that if $G_{i-1} : \bigcup_j I_{i-1}^j \to I_{i-1}$ is non-central, then I_{i+k} is a $\xi(k)$–scaled neighbourhood of I_{i+k+1}.*
(b) *For each $\xi > 0$ there is some $\hat{\xi} > 0$ such that if I_i is a ξ–scaled neighbourhood of I_{i+1} then I_{i+1} is a $\hat{\xi}$–scaled neighbourhood of any domain of G_{i+1}.*

This result gives us *real bounds* for some of our first return maps. We let $\chi := \xi(1) > 0$ from the above theorem for our map f.

The following theorem is an improvement of the classical Koebe Lemma. It is presented in more generality in [**SV**] as Proposition 2: 'a Koebe principle requiring less disjointness'. Note that actually for our purposes, the classical Koebe Lemma is enough.

Theorem 2.4. *Suppose that $g \in NF^2$. Then there exists a function $\nu : [0, |I|] \to [0, \infty)$ such that $\nu(\epsilon) \to 0$ as $\epsilon \to 0$ with the following properties. Suppose that for some intervals $J \subset T$ and a positive integer n we know that $g^n|_T$ is a diffeomorphism. Suppose further that $g^n(T)$ is a δ–scaled neighbourhood of $g^n(J)$ for some $\delta > 0$. Then,*

(a) *for every $x, y \in J$,*

$$\frac{|Dg^n(x)|}{|Dg^n(y)|} < \exp\left\{\nu(S(n,T))\sum_{i=0}^{n-1}|g^i(J)|\right\}\left[\frac{1+\delta}{\delta}\right]^2 =: C(\delta)$$

where $S(n,T) := \max_{0 \leqslant k \leqslant n-1} |f^k(T)|$.
(b) *T is a $\tilde{\delta}$–scaled neighbourhood of J whenever*

$$\tilde{\delta} := \frac{1}{2}\exp\{-\theta\}\left[\frac{1+\delta}{\delta}\right]^2\left(\frac{-2\theta + \delta(1-2\theta)}{2+\delta}\right)$$

is positive, where $\theta := \nu(S(n,T))\sum_{i=0}^{n-1}|g^i(J)|$.

Again we may use Lemma 2.2 to substitute $\nu(S(n,T))$ with $\nu'(|f^n(T)|)$ where we define $\nu'(|f^m(V)|) := \nu \circ \tau(|f^m(V)|)$. We will use the result of Theorem 2.3(b) extensively, but we use $\tilde{\delta}$ when $\theta = \nu'(|I_0|)$. Usually δ will be related to the χ we obtained following Theorem 2.3.

We will sometimes be in a situation where we wish to estimate the derivative of a function in between two points at which we know something about the derivative. The following two well known results allow us to do this. The following is known as the *Minimum Principle*; see, for example, Theorem IV.1.1 of [**MS**].

Theorem 2.5. *Let $T = [a, b] \subset I$ and $g : T \to g(T) \subset I$ be a C^1 diffeomorphism. Let $x \in (a, b)$. If for any $J^* \subset T^* \subset T$,*

$$B(g, T^*, J^*) > \mu_g > 0$$

then

$$|Dg(x)| > \mu_g^3 \min(|Dg(a)|, |Dg(b)|).$$

To see a proof of the following well known result see [**MS**].

Theorem 2.6. *For $g \in NF^2$ there exist $n_0 \in \mathbb{N}$ and $\rho_g > 1$ such that if p is a periodic point of period $n \geqslant n_0$ then $|Dg^n(p)| > \rho_g$.*

We are now ready to begin the proof of Theorem 1.2.

3. Well bounded case

Here we deal with the case where F_{i-2} is non–central and $f^j(T) \cap \partial I_{i+1} = \emptyset$ for all $0 \leqslant j < n_i$. In our estimates, we are principally interested in iterates of T landing in I_i^j for $j \neq 0$. By Theorem 2.3, the fact that F_{i-2} is non–central implies that the first return domains I_i^j are all well inside I_i. This enables us to estimate the sum for F_i, and is the reason we call this case well bounded.

Let $n_i' > n_{i+1}$ be minimal such that $f^{n_i'}(T) \subset I_i$. We will initially assume that we have some $\kappa > 0$ such that for the 'return sum',

$$\sum_{k=0}^{j_i} |F_i^k(f^{n_i'}(T))| < \kappa |f^{n_i}(T)| \tag{3}$$

where j_i is such that $F^{j_i}|_{f^{n_i'}(T)} = f^{n_i - n_i'}|_{f^{n_i'}(T)}$. We prove Proposition 1.3 before bounding this return sum in order to give an idea why we need bounds on return sums. Except for the proof of (3), this is similar to the proof of Lemma 5.3.4 of [**K1**]. There, it is assumed that $f \in C^3$ in order to bound the sum $\sum_{k=0}^{j_i-1} |F_i^k(f^{n_i'}(T))|$. Those methods fail in the C^2 case.

Proof of Proposition 1.3 assuming (3). Let $n_{i+1} = m_0 < m_1 < \cdots < m_{j_i} = n_i$ be all the integers between n_{i+1} and n_i such that $f^{m_j}(T) \subset I_i \setminus I_{i+1}$ for $j = 1, \ldots, j_i - 1$ and let $m_0 = n_{i+1}$. Now let $F_i : \bigcup_j U_i^j \to I_i$ be the first entry map to I_i. We will decompose $\sum_{k=1}^{n_i - n_{i+1}} |f^{k+n_{i+1}}(T)|$ as $\sum_{j=0}^{j_i-1} \sum_{k=1}^{m_{j+1}-m_j} |f^{k+m_j}(T)|$.

For $1 \leqslant j \leqslant j_i - 1$ and $1 \leqslant k < m_{j+1} - m_j$, let U_i^l be the domain of first entry to I_i such that $f^{m_j+k}(T) \subset U_i^l$. Suppose that $F_i|_{U_i^l} = f^{i_l}$. Then there exists an extension to $V_i^l \supset U_i^l$ so that $f^{i_l} : V_i^l \to I_{i-1}$ is a diffeomorphism.

Then by the Koebe Lemma we have the distortion bound: $\frac{|f^{k+m_j}(T))|}{|U_i^l|} \leqslant C(\chi)\frac{|f^{m_{j+1}}(T)|}{|I_i|}$. Whence

$$\sum_{k=1}^{m_{j+1}-m_j} |f^{m_j+k}(T)| \leqslant C(\chi)\left(\frac{|f^{m_{j+1}}(T)|}{|I_i|}\right)\sum_{k=0}^{m_{j+1}-m_j-1} |f^k(U_i^j)|$$
$$\leqslant C(\chi)\sigma_i\frac{|f^{m_{j+1}}(T)|}{|I_i|}.$$

Therefore

$$\sum_{k=1}^{n_i-n_{i+1}} |f^{k+n_{i+1}}(T)| \leqslant C(\chi)\frac{\sigma_i}{|I_i|}\sum_{j=1}^{j_i}|f^{m_j}(T)| = C(\chi)\frac{\sigma_i}{|I_i|}\sum_{k=0}^{j_i-1}|F_i^k(\hat{T})|$$

where $\hat{T} := f^{n_i'}(T)$. This is bounded above by $\kappa|f^{n_i}(T)|$ due to (3), so we are finished. $\square$

3.1. **Bounding return sums.** In this subsection we will introduce some tools which we use extensively in the remainder of this paper. We then use these tools to prove that (3) holds.

The proof of the following simple lemma is left to the reader.

Lemma 3.1. *For all $\delta > 0$ there exists $\Delta = \Delta(\delta) > 0$ such that $\Delta(\delta) \to 0$ as $\delta \to \infty$ with the following property. Suppose that U is an interval, $J \subset U$ is a subinterval and that the left and right components of $U \setminus J$ are denoted by L and R respectively. Suppose further that $|L|, |R| > \delta|J|$. Then*

$$B(U, J) < \Delta.$$

Let D_1 denote the set of non–central domains $F_i^{-1}(I_i)$, i.e. $D_1 = \bigcup_{j\neq 0} I_i^j$. Let D_2 denote the set of domains $F_i^{-1}(D_1)$ which are disjoint from the central domain. Inductively, we let D_k denote the set of domains $F_i^{-1}(D_{k-1})$ which are disjoint from the central domain. Then for any element $J_k \in D_k$, $F_i^k : J_k \to I_i$ is a diffeomorphism. We will bound $\sum_{j=0}^{k-1}|F_i^j(J_k)|$ for any $J_k \in D_k$ by showing that there exists some $\lambda < 1$ independent of i such that for $k > 1$ we have $B(I_i, J_k) \leqslant \lambda B(I_i, F_i(J_k))$. We let

$$\mu := \exp\{-\sigma'(|I_0|)\} \qquad (4)$$

where σ' is given by (2). By (1), if $J', f(J'), \ldots, f^m(J')$ is a disjoint set of intervals and $J \supset J'$, we have $B(f^m, J', J) > \mu$. Therefore, if $n(j)$ is the return time of I_i^j to I_i and $J \subset I_i^j$ then $B(f^{n(j)}, I_i^j, J) > \mu$.

The following lemma is Lemma 2.3 of [**GK**].

Lemma 3.2. *For every $\delta > 0$ there exists $\lambda' = \lambda'(\delta) < 1$ such that if $J \subset V \subset U$ are intervals and U is a δ–scaled neighbourhood of V then*

$$B(U, J) < \lambda' B(V, J).$$

Furthermore, $\lambda' \to 1$ as $\delta \to 0$.

We add this lemma to (1) as follows.

Lemma 3.3. *Given $\delta > 0$, there exist $0 < \lambda = \lambda(\delta) < 1$ and $\epsilon > 0$ such that if $|I_0| < \epsilon$ and I_{i-1} is a δ–scaled neighbourhood of I_i, then for any $J \subset I_i^j$ with $j \neq 0$,*

$$B(I_i, J) < \lambda B(I_i, F_i(J)).$$

Proof. From the previous lemma there exists some $\lambda' = \lambda'(\delta) < 1$ such that

$$B(I_i, J) < \lambda' B(I_i^j, J).$$

Now from (1) we obtain

$$B(I_i, J) < \lambda' \frac{B(I_i, F_i(J))}{\mu}$$

where μ is defined in (4). Since $\mu \to 1$ as $|I_0| \to 0$, if ϵ is chosen small enough then $\frac{\lambda'}{\mu} < 1$. We let $\lambda := \frac{\lambda'}{\mu}$. Thus $B(I_i, J) < \lambda B(I_i, F_i(J))$. □

We will consider $\lambda = \lambda(\tilde{\chi})$ where $\tilde{\chi}$ comes from Theorem 2.4(b) applied to χ and χ comes from Theorem 2.3(a), i.e. $\tilde{\chi}$ takes the role of δ in Lemma 3.3. In fact we shall adjust λ again in Section 5, but it will remain independent of i and strictly less than 1.

Proof of (3). For $k \geqslant 2$, $B(I_i, J_k) < \lambda^{k-1} B(I_i, F_i^{k-1}(J_k))$. Suppose that $F_i^{k-1}(J_k) \subset I_i^j$. Then by Lemma 3.1, using Theorems 2.3 and 2.4 (b), $B(I_i, I_i^j) < \Delta$ where $\Delta = \Delta(\tilde{\chi})$. Thus, it is easy to see $B(I_i, F_i^{k-1}(J_k)) < \Delta \frac{|F_i^{k-1}(J_k)|}{|I_i^j|}$. Now by the Koebe Lemma, $|F_i^{k-1}(J_k)| < C(\chi) |F_i^k(J_k)| \frac{|I_i^j|}{|I_i|}$, so we know that $B(I_i, F_i^{k-1}(J_k)) < C(\chi) \Delta \frac{|F_i^k(J_k)|}{|I_i|}$. We apply these estimates to the sizes of J_k:

$$|J_k| < \frac{|I_i|}{1 + \frac{2|I_i|}{\lambda^{k-1} C(\chi) \Delta |F_i^k(J_k)|}}.$$

Then $|J_k| < C \lambda^{k-1} |F_i^k(J_k)|$. So $\sum_{j=0}^{k-1} |F_i^j(J_k)| < C \frac{|F_i^k(J_k)|}{1-\lambda}$. Whence

$$\sum_{j=0}^{k} |F_i^j(J_k)| < |F_i^k(J_k)| \left(1 + \frac{C}{1-\lambda}\right).$$

This holds for any sum of returns which never lands in the central domain. It is independent of i. Letting $\kappa = \left(1 + \frac{C}{1-\lambda}\right)$ we prove (3). □

4. Cascade case

This section is devoted to the proof of Proposition 1.4. Note that if there is a uniform upper bound on the length of sequences $F_i, F_{i+1}, \ldots, F_{i+m}$ all having central returns then Theorem 2.3 implies that we may prove Proposition 1.4 as a well bounded case. However, there may be arbitrarily long sequences of consecutive central returns.

Proof of Proposition 1.4. We suppose that there i is such that $f^{n_i}(T) \subset I_i$ where F_{i-2} has a non–central return and F_{i+j} all have central returns for $j = 0, \ldots, m-1$ and that F_{i+m} has a non-central return. For $\xi > 0$ we will bound the sum

$$\sum_{k=1}^{n_i - n_{i+m+1}} |f^{k+n_{i+m+1}}(T)|^{1+\xi}.$$

For our intial estimates, we may omit ξ, but later it will be necessary to include it. Recall that we always assume here that $f^j(T) \cap \partial I_{i+m+1} = \emptyset$ for all $0 \leqslant j < n_i$.

Let $m_0 = n_{i+m+1}$ and let $m_0 < m_1 \leqslant n_i$ be the smallest integer such that $f^{m_1}(T) \subset I_i \setminus I_{i+1}$. Let $m_1 < m_2 \leqslant n_i$ be the next integer for which $f^{m_2}(T) \subset I_i \setminus I_{i+1}$ if such m_2 exists. Proceeding in this manner, we obtain a sequence, $n_{i+m+1} < m_1 < m_2 < \cdots < m_N = n_i$.

So

$$\sum_{k=1}^{n_i - n_{i+m+1}} |f^{k+n_{i+m+1}}(T)| = \sum_{j=0}^{N-1} \sum_{k=1}^{m_{j+1}-m_j} |f^{k+m_j}(T)|.$$

Define $m_{N-1} < m' \leqslant n_i$ to be minimal such that $f^{m'}(T) \subset I_i \setminus I_{i+m+1}$. Assuming that $F_i|_{I_i^0} = f^s$, there exists $0 \leqslant p \leqslant m$ such that $m' + sp = m_N = n_i$. We can rewrite the sum

$$\sum_{k=1}^{n_i - n_{i+m+1}} |f^{k+n_{i+m+1}}(T)| = \sum_{j=0}^{N-2} \sum_{k=1}^{m_{j+1}-m_j} |f^{k+m_j}(T)| + \sum_{k=1}^{m'-m_{N-1}} |f^{k+m_{N-1}}(T)|$$
$$+ \sum_{r=0}^{p-1} \sum_{k=1}^{s} |f^{k+rs+m'}(T)|.$$

Using the method from the proof of Proposition 1.3,

$$\sum_{k=1}^{m'} |f^{k+m_{N-1}}(T)| + \sum_{r=0}^{p-1} \sum_{k=1}^{s} |f^{k+rs+m'}(T)| \leqslant C(\chi) \frac{\sigma_{i,m}}{|I_i|} \sum_{r=0}^{p} |f^{rs+m'}(T)|.$$

We will deal with the sum on the right hand side later. We will first show that $\sum_{j=0}^{N-2} \sum_{k=1}^{m_{j+1}-m_j} |f^{k+m_j}(T)| \leqslant C\sigma_{i,m} \frac{|f^{m'}(T)|}{|I_i|}$.

We denote the left and right components of $I_j \setminus I_{j+1}$ by L_j and R_j respectively. We know from Theorem 2.3(a) and (b) that $\frac{|L_i|}{|I_{i+1}|}, \frac{|R_i|}{|I_{i+1}|} > \hat{\chi}$.

We define $\hat{F}_i : \bigcup_j \hat{I}_i^j \to I_i \setminus I_{i+1}$ to be the first return map to $I_i \setminus I_{i+1}$, such that $\hat{F}_i(\hat{I}_i^j) \in \{L_i, R_i\}$. As in the well bounded case, for each $1 \leqslant j \leqslant N-2$ and $1 \leqslant k < m_{j+1} - m_j$, there exists a first entry domain $\hat{U}$ to $I_i \setminus I_{i+1}$ such that $f^{k+m_j}(T) \subset \hat{U}$. We may assume that $f^{m_{j+1}-m_j-k}(\hat{U}) = L_i$. Indeed, for $1 \leqslant j \leqslant N-3$ there exists $\hat{I}_i^l$ such that $f^{m_{j+1}}(T) \subset \hat{I}_i^l \subset L_i$. We show that $\hat{I}_i^l$ is well inside L_i, which will allow us to estimate $\frac{|f^{k+m_j}(T)|}{|\hat{U}|}$.

Suppose that $F_i|_{\hat{I}_i^l} = f^{i_l}$. Then there exists an extension to $V_i^l \supset \hat{I}_i^l$ such that $f^{i_l} : V_i^l \to I_{i-1}$. Clearly $V_i^l \subset L_i$, otherwise niceness is contradicted. By Theorems 2.3(a) and 2.4(b), V_i^l (and thus L_i) is a $\tilde{\chi}$–scaled neighbourhood of $\hat{I}_i^l$.

For $1 \leqslant j \leqslant N-2$, we have $B(L_i, f^{m_{j+1}}(T)) > \mu B(\hat{U}, f^{k+m_j}(T))$ where μ is defined in (4). Therefore,

$$|f^{k+m_j}(T)| < \frac{|\hat{U}|}{1 + \frac{\mu}{B(L_i, f^{m_{j+1}}(T))}}.$$

As in the well bounded case, using a small adaptation of Lemma 3.3, replacing F_i by $\hat{F}_i$, we can show that $B(L_i, f^{m_j}(T)) < \lambda^{N-1-j} B(L_i, f^{m_{N-1}}(T))$ for $0 \leqslant j \leqslant N-2$. (Note that λ is still the $\lambda(\tilde{\chi})$ discussed following Lemma 3.3.) Therefore, it can be shown that

$$\sum_{j=0}^{N-2} \sum_{k=1}^{m_{j+1}-m_j} |f^{k+m_j}(T)| \leqslant \frac{C\sigma_{i,m}}{1-\lambda} B(L_i, f^{m_{N-1}}(T)).$$

But since $f^{m_{N-1}}(T) \subset I_i^{j'}$ for some $j' \neq 0$, we have

$$B(L_i, f^{m_{N-1}}(T)) < B(L_i, I_i^{j'}) \frac{|f^{m_{N-1}}(T)|}{|I_i^{j'}|}.$$

Notice that $F_i(f^{m_{N-1}}(T)) = f^{m'}(T)$. So the Koebe Lemma and Lemma 3.1 give $B(L_i, f^{m_{N-1}}(T)) < C(\chi)\Delta\frac{|f^{m'}(T)|}{|I_i|}$, whence

$$\sum_{j=0}^{N-2} \sum_{k=1}^{m_{j+1}-m_j} |f^{k+m_j}(T)| \leqslant C\sigma_{i,m} \frac{|f^{m'}(T)|}{|I_i|}.$$

It remains to bound $\sum_{r=0}^{p} |f^{rs+m'}(T)|^{1+\xi}$ (as can be seen below, we only really need $\xi > 0$ for our estimates in the low case). We assume that $f^{m'}(T) \cap \partial I_{i+j} \neq \emptyset$ for $1 \leqslant j < m$: otherwise we have $\sum_{r=0}^{p} |f^{rs+m'}(T)|^{1+\xi} < |I_i|^{1+\xi}$, and we are finished.

Let $\hat{T} = f^{m'}(J)$. There exists some $M \geqslant 0$ such that $F_i^M(\hat{T}) = f^{n_i}(T)$. We will bound $\sum_{k=0}^{M} |F_i^k(\hat{T})|^{1+\xi}$.

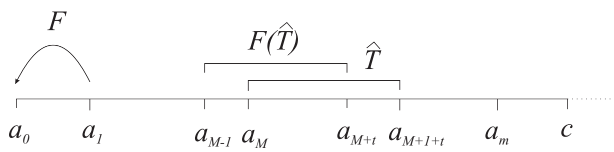

FIGURE 2. When $\hat{T}$ intersects the boundary points ∂I_j.

If M was uniformly bounded then we would be able to find some bound on $\sum_{k=0}^{M} |F_i^k(\hat{T})|$ easily. But M may be very large. We consider this sum in two cases: either F_i is high, or F_i is low (the high case is the most straightforward). For some background on this dichotomy see [**Ly**]. In both cases, we relabel $F_i|_{I_{i+1}}$ as F and I_i as I_0. Now let $I_k = (a_k, a_k')$. We are assuming that $F(c)$ is a maximum for F, see Figure 2.

The high case

We have two cases to consider. We first assume that F_j are high and central for $j = 0, \dots, m$. This is known as an Ulam–Neumann cascade.

Lemma 4.1. *In the high case,* $\sum_{k=0}^{M} |F^k(\hat{T})| < C|I_0|$.

Proof. We know that I_0 is a $\hat{\chi}$–scaled neighbourhood of I_1. We will use the Minimum Principle (Theorem 2.5) and Theorem 2.6 to estimate derivatives. The idea here is that either we have derivative uniformly greater than one in (a_1, a_m) and we can bound $\sum_{k=0}^{M} |F^k(\hat{T})|$ as a geometric sum; or we have a small derivative in some region, in which case we find a bound on the number of a_i that are in this region.

Let $\gamma > 1$ satisfy $\frac{\gamma}{\gamma-1} > \frac{1}{2\hat{\chi}}$. Then we may fix some integer $r \geqslant 1$ such that $2\hat{\chi} \sum_{i=0}^{r} \gamma^{-i} > 1$. Note that r only depends on $\hat{\chi}$. Observe that there is a fixed point $p \in (a_1, c)$. We can choose I_0 to be so small that the return time to it is greater than n_0 given in Theorem 2.6. Therefore, by that theorem, $|DF(p)| > \rho_f$. If $|DF(a_1)| \geqslant \gamma$ then from the Minimum Principle, $|DF|_{(a_1,p)} > \gamma'$ where $\gamma' = \mu^3 \min(\gamma, \rho_f)$ where μ is defined in terms of $|I_0|$ in (4). We fix I_0 to be small enough so that $\gamma' > 1$. Therefore, we have $\sum_{k=0}^{M} |F^k(\hat{T})| < \frac{\gamma'}{\gamma'-1} |F_i^M(\hat{T})|$.

Suppose now that there is some $u \in (a_1, c)$ such that $|DF|_{(a_1,u)} < \gamma$. We will show that this must mean that $u \in (a_1, a_r)$ and thus we can uniformly bound the sum of times that $\hat{T}$ lies in this region.

Suppose that $(a_1, a_s) \subset (a_1, u)$. Then we have $|a_{i+1} - a_i| > \frac{|a_i - a_{i-1}|}{\gamma}$ for all $i \leqslant s-1$. Therefore, if $(a_1, a_s) \subset U$ then

$$|c - a_0| > \sum_{i=0}^{s-1} |a_{i+1} - a_i| > |a_1 - a_0| \sum_{i=0}^{s} \gamma^{-i}.$$

We know that $|a_1 - a_0| > 2\hat{\chi}|c - a_0|$. By the definition of γ we must have $s \leqslant r$. Moreover, we have $|DF|_{(a_s,p)} > \gamma'$.

This helps us bound $\sum_{k=0}^{M} |F^k(\hat{T})|$ where $F^k(\hat{T}) \subset I_0 \setminus I_m$. We suppose that $F^M(\hat{T}) = (a_0, a_t)$ for $t \leqslant m$. See Figure 2. Then

$$\begin{aligned}\sum_{k=0}^{M} |F^k(\hat{T})| &= |a_1 - a_0| + \min(2, M-1)|a_2 - a_1| + \cdots \\ &\quad + \min(i, M-(i-1))|a_i - a_{t-i}| + \cdots + |a_{M+t} - a_{M+t-1}|.\end{aligned}$$

This is bounded above by

$$r|a_r - a_0| + |a_N - a_{N+1}| \sum_{i=0}^{\infty} \frac{\min(i, M-(i-1))}{\gamma'^i}.$$

The first summand is bounded by $r|I_0|$ and the second summand is bounded above by $C|a_N - a_{N+1}|$ for some $C > 0$. So we get $\sum_{k=0}^{M} |F_i^k(\hat{T})| < C|I_0|$ as required. □

The low case

We assume that we are in the same setting as above, but with F_0 central and low. This is known as a saddle node cascade. Again we would like to bound $\sum_{k=0}^{M} |F^k(\hat{T})|$ defined as above. However, as we shall see, we are only able to bound $\sum_{k=0}^{M} |F^k(\hat{T})|^{1+\xi}$.

Lemma 4.2. *In the low case,* $\sum_{k=0}^{M} |F^k(\hat{T})|^{1+\xi} < C|I_0|^{1+\xi}$.

Proof. We will apply the following result, a form of the Yoccoz Lemma, see for example [**FM**].

Lemma 4.3. *Suppose that $f \in NF^2$. Then for all $\delta, \delta' > 0$ there exists $C > 0$ such that if I_0 is a nice interval such that*

(1) *I_0 is a δ–scaled neighbourhood of I_1;*
(2) *F_k is low and central for $k = 0, \ldots, m$;*
(3) *there is some $0 < k < m$ with $\frac{|I_k|}{|I_{k+1}|} < 1 + \delta'$,*

then for $1 \leqslant k < m$,

$$\frac{1}{C} \frac{1}{\min(k, m-k)^2} < \frac{|I_{k-1} \setminus I_k|}{|I_0|} < \frac{C}{\min(k, m-k)^2}.$$

This lemma was suggested by Weixiao Shen. For the proof, see the appendix. (For comparison with other statements of the Yoccoz Lemma, note that we will prove that one consequence of our conditions for the lemma is that we have a lower bound on $\frac{|I_m \setminus I_{m+1}|}{|I_0|}$.)

Suppose that I_0 satisfies all the conditions of Lemma 4.3. In particular we assume that for some fixed $\delta' > 0$, we have $\frac{|I_k|}{|I_{k+1}|} < 1 + \delta'$ for some $0 < k < m$. Then for any $\xi > 0$,

$$\begin{aligned}
&\sum_{k=0}^{M} |F^k(\hat{T})|^{1+\xi} \\
&\quad < \sum_{k=0}^{m} \left(\frac{C|I_0|}{\min(k+t, m-(k+t))^2} + \cdots + \frac{C|I_0|}{\min(k+1, m-(k+1))^2} \right)^{1+\xi} \\
&\quad < C|I_0|^{1+\xi} \sum_{k=0}^{m} \left(\frac{1}{k+1} - \frac{1}{k+t} \right)^{1+\xi}.
\end{aligned}$$

The sum above is bounded above for any $\xi > 0$.

Next we suppose that the hypotheses of Lemma 4.3 do not hold. In particular, this means $\frac{|I_k|}{|I_{k+1}|} \geqslant 1+\delta'$ for $k = 0, \ldots, m$. Note that $|I_0| \geqslant (1+\delta')|I_1| \geqslant (1+\delta')^2|I_2| \geqslant \cdots \geqslant (1+\delta')^M |I_M|$. Therefore

$$\sum_{k=0}^{M} |F^k(\hat{T})| < \frac{1}{2} \sum_{k=0}^{M} k|I_k| \leqslant \frac{|I_0|}{2} \sum_{k=0}^{M} \frac{k}{(1+\delta')^k} < C|I_0|.$$

So the lemma is proved. $\square$

We have shown that in both low and high cases we have $\sum_{k=0}^{M} |F^k(\hat{T})|^{1+\xi} < C|I_0|^{1+\xi}$. We may apply the usual method to show that this means that $\sum_{k=1}^{n_i - m'} |f^{k+m'}(\hat{T})|^{1+\xi} < C\sigma_{i,m} \max_{m' < k \leqslant n_i} |f^k(T)|^{\xi}$. So there is some C_{casc} such that

$$\sum_{k=1}^{n_i - n_{i+m+1}} |f^{k+n_{i+m+1}}(T)|^{1+\xi} < C_{casc}\sigma_{i,m} \max_{n_{i+m} < k \leqslant n_i} |f^k(T)|^{\xi}$$

as required. $\square$

5. Exceptional case

In the last section we dealt completely with the saddle node cascade. It is easily shown, for example applying Lemma 5.1 below to all branches, that following a saddle node cascade we have a well bounded case, and so the conclusions of Proposition 1.3 hold. An Ulam–Neumann cascade, however, is not always followed by a well bounded case. We estimate the sum for F_i in this alternative case here. Most of the sum is dealt with using the

methods for the well bounded case, but we need some new techniques to deal with two of the branches of F_i.

We consider the sum for F_i where F_{i-2} has a central return and F_{i-1} has a high non–central return. The situation here is only slightly different from the case considered in Section 3, since we can prove that all domains of F_i are well inside I_i, except possibly two. Both of these domains I_i^j have $F_i|_{I_i^j} = F_{i-1}|_{I_i^j}$. We denote the left–hand such interval by I_i^L and the right–hand one by I_i^R, see Figure 3. These are the exceptional domains. If I_{i-1} is

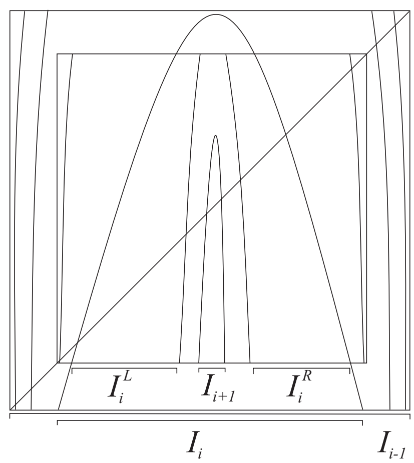

FIGURE 3. The exceptional case.

a $\hat{\chi}$-scaled neighbourhood of I_i then by Theorem 2.3 we know that I_i is a $\tilde{\hat{\chi}}$–scaled neighbourhood of both I_i^L and I_i^R, and we may proceed as in the well bounded case. But this will not always be so if I_{i-1} is at the end of a long Ulam–Neumann cascade. So we will assume that I_{i-1} is not a $\tilde{\hat{\chi}}$-scaled neighbourhood of I_i. Without loss of generality, we suppose that $F_{i-1}(c)$ is a maximum for $F_{i-1} : I_i \to I_{i-1}$.

We are now ready to begin the proof of Proposition 1.5. The strategy for the proof is as follows.

- Show there is some upper bound on $B(I_i, I_i^j)$ for $j \neq L, R$.
- State our main result in the proof: Proposition 5.3. We suppose that we have some interval $J \subset I_i^j$ for $j \neq L, R, 0$; $F_i(J), \ldots, F_i^m(J) \subset I_i^L \cup I_i^R$; and $F_i^{m+1}(J) \subset I_i^{j'}$ for $j' \neq L, R, 0$. Then there exists some $\lambda < 1$ such that $B(I_i, J) < \lambda B(I_i, F_i^{m+1}(J))$. Furthermore, $\sum_{k=1}^m |F_i^k(J)| < B(I_i, F_i^{m+1}(J))|I_i|$. We are then able to prove Proposition 1.5. In the rest of this section we prove Proposition 5.3: essentially we need an upper bound on $\sum_{k=1}^m |F_i^k(J)|$.

- In Lemma 5.4 we show that there exist an interval $V \subset I_i$ and $\gamma > 1$ such that
$$|DF_i|_{(I_i^L \cup I_i^R)\setminus V} > \gamma.$$
This allows us to bound parts of the sum $\sum_{k=1}^m |F_i^k(J)|$ which lie in $(I_i^L \cup I_i^R) \setminus V$.
- We next focus on V. We take first return maps to V and use decay of cross–ratios again to estimate sums of intervals in V, see Lemma 5.6. We can then complete the proof of Proposition 5.3

We first show in the following simple lemma that we have uniform bounds on how deep the domains of F_i are in I_i for all domains except I_i^L, I_i^R.

Lemma 5.1. *In the exceptional case outlined above, if $j \neq L, 0, R$ then I_i is a $\tilde{\hat{\chi}}$–scaled neighbourhood of I_i^j.*

In fact, a similar result holds for the central domain too by Theorem 2.3, but this is not important for us here. This lemma proves that we can treat the case where F_{i-2} is central and F_{i-1} is low and non–central as a well bounded case.

As we shall see, the proof of this lemma is reminiscent of the cascade case since we follow iterates of intervals along the central branch of some $F_{i'}$.

Proof. There exists some maximal $i' < i$ such that $F_{i'-2}$ is non–central. Then by Theorem 2.3, $I_{i'}$ is a $\hat{\chi}$–scaled neighbourhood of $I_{i'+1}$.

For $j \neq L, R$ we will find $F_i|_{I_i^j}$ as a composition of some branches of $F_{i'}$ in order to find some extensions. $F_{i'}|_{I_{i'+1}}$ maps I_i^j out of I_i along the cascade, through the sets $I_{i-1} \setminus I_i$, $I_{i-2} \setminus I_{i-1}$ and so on, until it maps to some interval in $I_{i'+1} \setminus I_{i'+2}$. Then this interval is mapped into some $I_{i'}^{j'}$. This then maps back into $I_{i'+1}$. The process may be repeated many times before I_i^j is finally mapped back to I_i.

So know that $F_i|_{I_i^j}$ is a composition of maps as follows. Let $j_1 \neq 0$ be such that $(F_{i'}^{i-i'}|_{I_{i'+1}})(I_i^j) \subset I_{i'}^{j_1}$. Let $k_1 = i - i'$. If $F_i|_{I_i^j} = (F_{i'}|_{I_{i'}^{j_1}})(F_{i'}^{(i-i')}|_{I_{i'+1}})|_{I_i^j}$ then we stop here; we say $r = 1$. Otherwise, let $k_2 \geqslant 0$ be minimal such that $F_{i'}^{k_1+1+k_2}(I_i^j) \subset I_{i'} \setminus I_{i'+1}$. Let $j_2 \neq 0$ be such that $F_{i'}^{k_1+1+k_2}(I_i^j) \subset I_{i'}^{j_2}$. If $F_i|_{I_i^j} = F_{i'}^{k_1+1+k_2+1}|_{I_i^j}$ then we stop here; we say $r = 2$. Otherwise, we continue this process until we finally return to I_i and obtain k_r.

Suppose that $r = 1$. That is,
$$F_i|_{I_i^j} = F_{i'}^{(i-i')+1}|_{I_i^j}.$$
Let U denote $F_{i'}^{(i-i')}(I_i^j)$ and U' denote $I_{i'}^{j_1}$. Then $F_{i'}(U) = I_i$ and $F_{i'}(U') = I_{i'}$. We know that $I_{i'}$ is a $\hat{\chi}$–scaled neighbourhood of I_i. So if we can show

that, taking the appropriate branch, $(F_{i'}^{-(i-i')}|_{I_{i'+1}})(U') \subset I_i$, we know by Theorem 2.4(b) that I_i is a $\tilde{\hat{\chi}}$–scaled neighbourhood of I_i^j (since all the intervals we are concerned with are disjoint). It is easy to see that for this branch, $(F_{i'}^{-(i-i')}|_{I_{i'+1}})(U') \subset I_i$ by the structure of the saddle node cascade since we have $(F_{i'}^{-1}|_{I_{i'+1}})(U') \subset I_{i'+1} \setminus I_{i'+2}$, $(F_{i'}^{-2}|_{I_{i'+1}})(U') \subset I_{i'+2} \setminus I_{i'+3}$ and so on. So the lemma is proved when $r = 1$.

In the more general case, where $r > 1$ and

$$F_i|_{I_i^j} = F_{i'}^{\sum_{l=1}^r (k_l+1)}|_{I_i^j}$$

we may apply the same idea, again using the disjointness of the domains of the first return map, to prove that I_i is a $\tilde{\hat{\chi}}$–scaled neighbourhood of I_i^j. $\square$

If necessary we adjust λ so that $\lambda(\tilde{\hat{\chi}}) \leqslant \lambda < 1$.

By the above, if I_i is a $\tilde{\hat{\chi}}$–scaled neighbourhood of I_i^L and I_i^R then we can proceed with the method in the well bounded case to prove Proposition 1.5. But this is not generally the case. So for our work here, we may assume that I_i is not a $\tilde{\hat{\chi}}$–scaled neighbourhood of I_i^L or I_i^R, and that some iterate of J enters $I_i^L \cup I_i^R$.

Remark 5.2. In the previous sections we had uniform upper bounds on the cross–ratio $B(I_i, I_i^j)$ for all j and so we obtained estimates on the decay of cross–ratios directly. This was used to estimate the sums of intervals. The problem we often encounter in this section is that sometimes we only get good estimates on how cross–ratios decay and sometimes we only get good estimates for the decay of the sizes of intervals. But these estimates are difficult to marry together directly, so we will have to split up such cases. The process is first described in the proof of Proposition 1.5 and again in the proof of Lemma 5.6. (As we will see later, this splitting scheme deals with the cases where we enter $I_i^L \cup I_i^R$ from I_i; V from $I_i^L \cup I_i^R$; and Λ from V.)

The principal result in this section is the following proposition.

Proposition 5.3. *If $J, F_i(J), \ldots, F_i^m(J) \subset I_i^L \cup I_i^R$ then*

1. *there exists some $0 \leqslant \hat{m} < m$ such that $\sum_{k=0}^m |F_i^k(J)| < C(|F_i^m(J)| + |F_i^{\hat{m}}(J)|)$;*
2. *for some $\lambda < 1$ independent of i, if $F_i^{m+1}(J) \subset I_i^j$, $j \neq L, 0, R$ then*
   (a) $\sum_{k=0}^m |F_i^k(J)| < CB(I_i, F_i^{m+1}(J))|I_i|$;
   (b) *letting J' be the element of $F_i^{-1}(J)$ inside some interval $I^{j'}$ for $j' \neq L, 0, R$ then we have $B(I_i, J') < \lambda B(I_i, F_i^{m+2}(J'))$.*

See Figure 4 for a schematic representation of the situation of this proposition. If necessary we will adjust the $\lambda < 1$ we use throughout this paper so that we may assume that the proposition above holds for that λ.

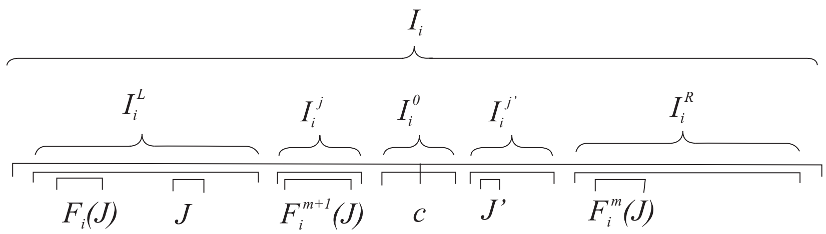

FIGURE 4. An illustration of Proposition 5.3.

Proof of Proposition 1.5 assuming Proposition 5.3. As in the proof in the well bounded case, we first show that we are principally concerned with the intervals inside I_i. Again, the proof of this fact is a slight modified version of the proof in the well bounded case.

Let $n_{i+1} < m_1 < \cdots < m_{j_i} = n_i$ be all the integers between n_{i+1} and n_i such that $f^{m_j}(T) \subset I_i \setminus I_{i+1}$ for $j = 1, \ldots, j_i - 1$ and let $m_0 = n_{i+1}$. Let $F_i : \bigcup_i U_i^j \to I_i$ be the first entry map to I_i. As before, we will decompose the sum $\sum_{i=n_{i+1}+1}^{n_i} |f^i(T)|$ as $\sum_{j=0}^{j_i-1} \sum_{k=1}^{m_{j+1}-m_j} |f^{m_j+k}(T)|$.

Suppose that $f^{m_j+1}(T) \subset U_i^j$ for some U_i^j. Suppose further, that $F_i|_{U_i^j} = f^{i_j}$. Then there exists an extension to $V_i^j \supset U_i^j$ so that $f^{i_j} : V_i^j \to I_{i'-1}$ is a diffeomorphism, where i' is defined in the proof of Lemma 5.1. Then we have distortion bounds as usual: $\frac{|f^k(f^{m_j+1}(T))|}{|f^k(U_i^j)|} \leqslant C(\chi)\frac{|f^{m_j+1}(T)|}{|I_i|}$. Thus,

$$\sum_{k=1}^{m_{j+1}-m_j} |f^{m_j+k}(T)| < C(\chi)\sigma_i \frac{|f^{m_{j+1}}(T)|}{|I_i|}.$$

Therefore, $\sum_{j=n_{i+1}+1}^{n_i} |f^i(T)| < C(\chi)\frac{\sigma_i}{|I_i|}\sum_{j=1}^{j_i} |f^{m_i}(T)|$. I.e. we are principally interested in the sum $\sum_{j=1}^{j_i} |f^{m_i}(T)|$, that is $\sum_{k=0}^{j_i-1} |F_i^k(\hat{T})|$ where $\hat{T} = f^{m_1}(T)$. In fact, we focus on bounding $\sum_{k=0}^{j_i-2} |F_i^k(\hat{T})|$.

We split $\hat{T}, F_i(\hat{T}), \ldots, F_i^{j_i-2}(\hat{T})$ into two groups: one for those intervals outside $I_i^L \cup I_i^R$ and one for those inside $I_i^L \cup I_i^R$. Suppose that J is an interval such that for some $k \geqslant 0$, we have $F_i^k(J) \subset I_i^j$ for some $j \neq L, 0, R$; then $F_i^{k+1}(J), F_i^{k+2}(J), \ldots, F_i^{k'}(J) \subset I_i^L \cup I_i^R$ for some $k' > k$; and finally $F_i^{k'+1}(J) \subset I_i^{j'}$ for some $j' \neq L, 0, R$. From the last part of Proposition 5.3 we have

$$B(I_i, F_i^k(J)) < \lambda B(I_i, F_i^{k'+1}(J)).$$

Therefore, we can bound the sums of intervals which lie in the intervals I_i^j for all $j \neq L, R$ in a similar manner to that for the well bounded case, independently of those intervals inside $I_i^L \cup I_i^R$, as follows.

Given $k \geqslant 0$ such that $F_i^k(\hat{T}) \subset I_i^j$ for some $j \neq L, 0, R$ we wish to estimate $|F_i^k(\hat{T})|$. Let $0 \leqslant \hat{k} \leqslant j_i - 2$ be maximal such that $F_i^{\hat{k}}(\hat{T}) \subset I_i^{j'}$ for some $j' \neq L, R$. Then we apply Proposition 5.3 repeatedly to obtain $B(I_i, F_i^k(\hat{T})) < \lambda^l B(I_i, F_i^{\hat{k}}(\hat{T}))$ for some $l \geqslant 0$. The l counts the number of times that $F_i^{k+r}(\hat{T})$ lies outside $I_i^L \cup I_i^R$ for $0 < r \leqslant \hat{k}$. Then

$$|F_i^k(\hat{T})| < \frac{|I_i|}{1 + \frac{2}{\lambda^l B(I_i, F_i^{\hat{k}}(\hat{T}))}}.$$

We have two cases. In the first case we have $\hat{k} = j_i - 2$. Then

$$\begin{aligned} B(I_i, F_i^{j_i-2}(\hat{T})) &< B(I_i, I_i^{j'}) \frac{|F_i^{j_i-2}(\hat{T})|}{|I_i^{j'}|} < \Delta(\tilde{\hat{\chi}}) \frac{|F_i^{j_i-2}(\hat{T})|}{|I_i^{j'}|} \\ &< \Delta(\tilde{\hat{\chi}}) C(\tilde{\hat{\chi}}) |F_i^{j_i-1}(\hat{T})|. \end{aligned}$$

Therefore, $|F_i^k(\hat{T})| < C\lambda^l |F_i^{j_i-1}(\hat{T})|$. This suffices to prove an upper bound of the form $C|F_i^{j_i-1}(\hat{T})|$ for the F_i–iterates of $\hat{T}$ outside $I_i^L \cup I_i^R$ in this case.

In the second case $\hat{k} < j_i - 2$. We have

$$B(I_i, F_i^{\hat{k}}(\hat{T})) < B(I_i, I_i^{j'}) \frac{|F_i^{\hat{k}}(\hat{T})|}{|I_i^{j'}|} < \frac{\Delta |F_i^{\hat{k}}(\hat{T})|}{|I_i^{j'}|}.$$

Since $|F_i^{\hat{k}}(\hat{T})| < C(\chi) |F_i^{\hat{k}+1}(\hat{T})| \frac{|I_i^{j'}|}{|I_i|}$. Therefore, in this case we have a bound of the form $C|F_i^{\hat{k}+1}(\hat{T})|$ for the iterates of T outside $I_i^L \cup I_i^R$.

Finally we use the above information about sizes of intervals outside $I_i^L \cup I_i^R$ to bound the sums of intervals inside $I_i^L \cup I_i^R$ too. In the first case above, we have a bound of the form $C|F_i^{j_i-1}(\hat{T})|$ for the iterates of T in $I_i^L \cup I_i^R$. In the second case above, we have a bound of the form $C(|F_i^{\hat{k}}(\hat{T})| + |F_i^{\hat{m}}(\hat{T})| + |F_i^{j_i-1}(\hat{T})|)$ for the iterates of T in $I_i^L \cup I_i^R$.

So in the worst case we have the bound

$$C_{ex}\sigma_i \left(\frac{|f^{n_i}(T)|}{|I_i|} + \frac{|f^{n_i,2}(T)|}{|I_i|} + \frac{|f^{n_i,3}(T)|}{|I_i|} \right)$$

for the sum $\sum_{k=n_{i+1}+1}^{n_i} |f^k(T)|$, as required. $\square$

5.1. **Proof of Proposition 5.3.** Denote the smallest interval containing both I_i^L and I_i^R by I_i'. Recall that we are assuming that the critical point is a maximum for $F_{i-1}|_{I_i'}$. (Recall that $F_i|_{I_i^L \cup I_i^R} = F_{i-1}|_{I_i^L \cup I_i^R}$.) This means

that there is some fixed point p of F_i in I_i^R. Clearly, there also exists a point $p' \in I_i^L$ such that $F_i(p') = p$. Let $V := (p', p)$.

We outline the proof of Proposition 5.3 as follows. We suppose that some iterate of J enters V. Let $0 \leqslant s_1 \leqslant s_2 \leqslant s_3$ be defined as follows. $F_i^k(J) \subset I_i' \setminus V$ for $1 \leqslant k \leqslant s_1$; $F_i^{s_1+1}(J) \subset V \cap (I_i^L \cup I_i^R)$; and $F_i^{s_2+k}(J) \subset I_i' \setminus V$ for $1 \leqslant k \leqslant s_3 - s_2$. Any sum of the form $\sum_{k=0}^m |F_i^k(J)|$ can be broken up into blocks consisting of such sums.

The scheme for proving Proposition 5.3 is to firstly to show that $|DF_i|_{I_i' \setminus V}$ is uniformly large. This is proved in Lemma 5.4 and helps to deal with the sums $\sum_{k=0}^{s_1} |F_i^k(J)|$ and $\sum_{k=1}^{s_3-s_2} |F_i^{s_2+k}(J)|$. Then we have to prove that we have bounds on the sums of intervals which return to V. This, proved in Lemma 5.6, helps to deal with $\sum_{k=1}^{s_2-s_1} |F_i^{s_1+k}(J)|$.

Note that the proof of Proposition 5.3 is the only time in this paper that we use the symmetry of the map (and it is only a simplifying assumption).

Lemma 5.4. *There exists some $\gamma > 1$ independent of i such that*

$$|DF_i|_{I_i' \setminus V} > \gamma.$$

Proof. We start by observing as in the last section that $|DF_i(p)| > \rho_f$. By symmetry, $|DF_i(p')| > \rho_f$ too. Observe that I_i^L also contains a fixed point q of F_i. We have $|DF_i(q)| > \rho_f$ too. Furthermore, there exists a point $q' \in I_i^R$ such that $F_i(q') = q$. From symmetry, $|DF_i(q')| > \rho_f$.

We can estimate $|DF_i|_{(p,q')}$ using the Minimum Principle as follows. We use our μ given in (4) in place of μ_g. Then $|DF_i|_{(p,q')} > \mu^3 \rho_f$. When I_0 is small enough, μ is close to 1. Thus we may ensure that our intervals are so small that $|DF_i|_{(p,q')} > \rho$ for some $\rho > 1$. (To fix precisely how small our intervals must be, we can, for example, choose $\rho = \sqrt{\rho_f}$.) By symmetry, $|DF_i|_{(q,p')} > \rho$.

We deal with the remaining part of the proof of the lemma by showing that F_i has large derivative when $x \in I_i'$ and either $x < q$ or $x > q'$. We use the following consequence of Theorem 2.3 and the Minimum Principle.

Claim. *There exists some $\gamma' = \gamma'(\chi) > 1$ such that, denoting $I_i^L = (l^-, l^+)$ and $I_i^R = (r^-, r^+)$, if I_0 is sufficiently small and $B(I_i, I_i^L), B(I_i, I_i^R)$ are sufficiently large then*

$$|DF_i|_{(l^-,q)}, |DF_i|_{(q',r^+)} > \gamma'.$$

Proof. Let $\theta := \frac{1}{2}\left(\frac{|I_{i'}|}{|I_{i'+1}|} - 1\right) > \hat{\chi}$ where i' is defined in the proof of Lemma 5.1. We suppose that $|DF_{i'}|_{I_{i'+1} \setminus I_i} \leqslant 1 + 2\theta$. Then we prove by

induction that $\frac{|I_{i'+k}|}{|I_{i'+k+1}|} \geqslant 1+2\theta$ for $0 \leqslant k < i-i'$. By construction it is true for $k=0$. We assume that it is true for some $0 \leqslant k < i-i'-1$. Then

$$\begin{aligned}\frac{|I_{i'+k+1}|}{|I_{i'+k+2}|} &\geqslant \frac{|I_{i'+k+2}| + (\sup_{I_{i'+1}\setminus I_i} |DF_{i'}|)^{-1}|I_{i'+k} \setminus I_{i'+k+1}|}{|I_{i'+k+2}|} \\ &\geqslant 1 + \left(\frac{2\theta}{1+2\theta}\right)\frac{|I_{i'+k+1}|}{|I_{i'+k+2}|}\end{aligned}$$

Then it is easy to see that $\frac{|I_{i'+k+1}|}{|I_{i'+k+2}|} \geqslant 1+2\theta$ as required.

In particular, we have proved that $|DF_{i'}|_{I_{i'+1}\setminus I_i} \leqslant 1+2\theta$ implies that I_i is a $\tilde{\theta}$–scaled neighbourhood of both I_i^L and I_i^R: a contradiction (since $\tilde{\theta} > \tilde{\hat{\chi}}$). So there must exist some $x \in I_{i'+1}\setminus I_i$ such that $|DF_i(x)| \geqslant 1+2\theta > 1+2\hat{\chi}$. Therefore, by Theorem 2.5 and (1) we have

$$|DF_{i'}|_{(x_0,p)} > \mu^3 \min(1+2\hat{\chi}, \rho_f).$$

Choosing $|I_0|$ small we have some $\gamma' > 1$ such that $|DF_i|_{(x_0,q)} > \gamma'$. In particular $|DF_i|_{(l^-,q)} > \gamma'$. Similarly we can show $|DF_i|_{(q',r^+)} > \gamma'$. □

Letting $\gamma := \min(\rho, \gamma')$, the lemma is proved. □

By the above, we will be able to estimate the sizes of iterates of T inside $(I_i^L \cup I_i^R)\setminus V$ as a geometric sum.

We will need some real bounds for V. The following lemma, which contrasts with Lemma 5.4, will later be used to obtain these bounds.

Lemma 5.5. *There exists some $\hat{C} = \hat{C}(\chi, |I_i'|) > 0$, where $\hat{C}(\chi, |I_i'|)$ tends to some constant $\hat{C}(\chi)$ as $|I_i'| \to 0$, such that*

$$|DF_i|_{I_i^L\cup I_i^R} < \hat{C}.$$

Proof. We work with $F_{i'} : I_{i'+1} \to I_{i'}$ where i' is defined in the proof of Lemma 5.1. There exists some $m \geqslant 1$ such that $F_{i'}|_{I_{i'+1}} = f^m|_{I_{i'+1}}$. We can decompose this map into two maps so that $F_{i'} = L \circ g$ where $g = f|_{U_\phi}$, i.e $g(x) = f(c) - |\phi(x)|^\alpha$, and $L = f^{m-1} : f(I_{i'+1}) \to I_{i'}$.

By Theorems 2.4(a) and 2.3(a) we have $\frac{DL(x)}{DL(y)} < C(\chi)$ for $x, y \in f(I_{i+1})$. So

$$|DL(x)| \leqslant C(\chi)\frac{|I_i|}{|f(I_{i+1})|} = C(\chi)\frac{|I_i|}{\left|\phi\left(\frac{|I_{i+1}|}{2}\right)^\alpha\right|}$$

for $x \in f(I_{i+1})$. Also

$$|Dg(x)| = \alpha|D\phi(x)||\phi(x)^{\alpha-1}| < \alpha \sup_{x\in I_{i'+1}} |D\phi(x)| \left|\phi\left(\frac{|I_{i+1}|}{2}\right)\right|^{\alpha-1}.$$

For $\hat{U} \subset U_\phi$ a small neighbourhood of c, let $\mathrm{Dist}(\phi, \hat{U}) := \sup_{x,y\in\hat{U}} \frac{|D\phi(x)|}{|D\phi(y)|}$. Observe that as I_i' becomes smaller, $\mathrm{Dist}(\phi, I_i')$ tends to 1. For $x \in I_i^L \cup I_i^R$,

$$|DF_i(x)| < \alpha C(\chi) \frac{\sup_{x\in I_{i+1}} |D\phi(x)||I_i|}{\left|\phi\left(\frac{|I_{i+1}|}{2}\right)\right|} < 2\alpha C(\chi)\mathrm{Dist}(\phi, I_i')\frac{|I_i|}{|I_{i+1}|}.$$

Since we have assumed that $\frac{|I_i|}{|I_{i+1}|}$ is bounded below, there is some constant $C > 0$ such that for all $x \in I_i'$,

$$|DF_i(x)| < CC(\chi)\mathrm{Dist}(\phi, I_i').$$

Letting $\hat{C}(\chi, |I_i'|) := CC(\chi)\mathrm{Dist}(\phi, I_i')$ we have proved the lemma. □

We denote the first return map to V by $\hat{F}_i : \bigcup_j V^j \to V$. We first wish to find some control on the sizes of the domains of $\hat{F}_i$. Let $m_{V,j}$ be such that $\hat{F}_i|_{V^j} = F_i^{m_{V,j}}|_{V^j}$. The following lemma is key to proving Proposition 5.3.

Lemma 5.6. *If $F_i^{l_1}(J), \ldots, F_i^{l_m}(J) \subset V \cap (I_i^L \cup I_i^R)$ are all the iterates of J up to l_m which lie in $V \cap (I_i^L \cup I_i^R)$, and all intermediate iterates $F_i^k(J)$ for $k = 0, 1, \ldots, l_m$ lie in $I_i^L \cup I_i^R$ then*

$$\sum_{k=0}^{l_m} |F_i^k(J)| < C|F_i^{l_m}(J)|.$$

Furthermore, there exists $\lambda_V < 1$ such that $|J| < C\lambda_V^{l_m - m}|F_i^{l_m}(J)|$.

Proof. We split the sum as follows

$$\sum_{k=0}^{l_m} |F_i^k(J)| = \sum_{j=0}^{m-1} \sum_{k=1}^{l_{j+1}-l_j} |F_i^{l_j+k}(J)|$$

where we let $l_0 = -1$. We know from Lemma 5.4 that $|DF_i|_{I_i'\setminus V} > \gamma$ so

$$\sum_{k=1}^{l_{j+1}-l_j} |F_i^{l_j+k}(J)| < |F_i^{l_{j+1}}(J)| \sum_{k=0}^{l_{j+1}-l_j-1} \gamma^{-k} < \frac{|F_i^{l_{j+1}}(J)|}{1-\gamma^{-1}}.$$

Whence,

$$\sum_{k=0}^{l_m} |F_i^k(J)| < \frac{1}{1-\gamma^{-1}} \sum_{j=0}^{m} |F_i^{l_j}(J)|.$$

So we only need bound the sum of returns to V.

Denote the rightmost element of $\bigcup^j V^j$ by V^1 and the leftmost element by V^2 (observe that $\hat{F}_i|_{V^1} = F_i^2|_{V^1}$ and $\hat{F}_i|_{V^2} = F_i^2|_{V^2}$). We get an estimate on how deep each V^j is inside V for $j > 2$ because V^1 and V^2 have some definite size compared to $|V|$; since by Lemma 5.5 we know that $|V^1|, |V^2| > \frac{|V|}{\hat{C}^2}$. Therefore, there exists some δ_0' depending only on f such that V is a δ_0'–scaled neighbourhood of V^j for all $j > 2$. So by Lemma 3.2, there exists some $\lambda_V' < 1$ depending on δ_0' such that for any interval $J' \subset V^j$,

$B(V,J') < \lambda'_V B(V^j,J')$ for $j>2$ (in fact this is also shown in the claim below). As usual we can use Lemma 3.3 to conclude that there exists some $\lambda_V<1$ such that $B(V,J') < \lambda_V B(V,\hat{F}_i(J'))$. If we remain away from V^1 and V^2, this fact and the usual argument would be sufficient to obtain the required bound on sums.

We must deal with the case where iterates enter $V^1,\ V^2$. The idea is to split the situation into the case where intervals land in a region where $|D\hat{F}_i|$ is large and the case when the intervals land in a region where we don't have good estimates on $|D\hat{F}_i|$.

We first focus on V^2. We know from Theorem 2.6 that $|DF_i(p')| > \rho_f$ and so $|D\hat{F}_i(p')| > \rho_f^2$. There must also exist some fixed point r of $\hat{F}_i$ in V^2 with $|D\hat{F}_i(r)| > \rho_f$. Letting $\Lambda_2 := (p',r)$ and applying the Minimum Principle as before, we obtain $|D\hat{F}_i|_{\Lambda_2} > \rho$ for some $\rho>1$. Let r' be the point in V^1 such that $\hat{F}_i(r')=r$. Then adjusting $\rho>1$ if necessary, $|D\hat{F}_i|_{(r',p)} > \rho$. We define Λ_1 to be the interval in V^1 which has $\hat{F}_i(\Lambda_1) = V\setminus V^2$. Clearly $\Lambda_1 \subset (r',p)$, so $|D\hat{F}_i|_{\Lambda_1} > \rho$. For convenience later, we let $\Lambda := \Lambda_1\cup\Lambda_2$.

We are now ready to deal with bounding $\sum_{k=0}^{m-1}|\hat{F}_i^{\,k}(J)|$. Observe that $\hat{F}_i^{\,m-1}(J)$ must be contained in some V^j. Suppose first that $j>2$; we deal with the case where $j=1$ or 2 later. Suppose further that $J\subset V^{j'}$ and $j'>2$; here the other case is similar. We will again split up the sum. Let $N_0'=0$. Let N_1 be minimal such that $\hat{F}_i^{\,N_1}(J)\cap\Lambda=\emptyset$ and $\hat{F}_i^{\,N_1+1}(J)\subset\Lambda$. Let $N_1'>N_1$ be minimal such that $\hat{F}_i^{\,N_1'}(J)\subset\Lambda$ and $\hat{F}_i^{\,N_1'+1}(J)\cap\Lambda=\emptyset$. In this way we obtain $N_0'<N_1<N_1'<\cdots<N_{M-1}<N_{M-1}'$ so that

$$\begin{aligned}\sum_{k=0}^{m-1}|\hat{F}_i^{\,k}(J)| &= \sum_{j=0}^{M-1}\left(\sum_{k=1}^{N_{j+1}-N_j'}|\hat{F}_i^{\,N_j'+k}(J)| + \sum_{k=1}^{N_{j+1}'-N_{j+1}}|\hat{F}_i^{\,N_{j+1}+k}(J)|\right)\\ &\quad+\sum_{k=1}^{N_M-N_{M-1}'}|\hat{F}_i^{\,N_{M-1}'+k}(J)|\end{aligned}$$

where $N_M=m-1$. Observe that the first sum in the brackets concerns intervals which land inside Λ and the second sum in the brackets concerns intervals in $V\setminus\Lambda$. Then

$$\sum_{k=1}^{N_{j+1}'-N_{j+1}}|\hat{F}_i^{\,N_{j+1}+k}(J)| < |\hat{F}_i^{\,N_{j+1}'}(J)|\sum_{k=0}^{N_{j+1}-N_{j+1}'-1}\rho^{-k} < \frac{C}{1-\rho^{-1}}|\hat{F}_i^{\,N_{j+1}'}(J)|$$

for some C.

Now we consider $\sum_{k=1}^{N_{j+1}-N_j'}|\hat{F}_i^{\,N_j'+k}(J)|$. In fact we learn most from estimating the sum $\sum_{k=1}^{N_M-N_{M-1}'}|\hat{F}_i^{\,N_{M-1}'+k}(J)|$. If necessary we make $\lambda_V<1$ smaller so that for $J\subset V^j\setminus\Lambda_j$ for $j=1,2$ we have $B(V,J)<\lambda_V B(V,F_i(J))$. Then

for $1 \leqslant k < N_m - N'_{M-1}$,

$$B(V, \hat{F}_i^{N'_{M-1}+k}(J)) < \lambda_V^{N_M - N'_{M-1}-k} B(V, \hat{F}_i^{N_M}(J)).$$

Recalling that $M = m-1$ we calculate $B(V, \hat{F}_i^{m-1}(J)) < B(V, V^j)\frac{|\hat{F}_i^{m-1}(J)|}{|V^j|}$. Letting $B_V := \max\left\{\sup_{j>2} B(V, V^j), B(V, V^1 \setminus \Lambda_1), B(V, V^2 \setminus \Lambda_2)\right\}$, we obtain

$$|\hat{F}_i^{N'_{M-1}+k}(J)| < \frac{|V|}{1 + \frac{2|V^j|}{\lambda_V^{N_M - N'_{M-1}-k} B_V |\hat{F}_i^{m-1}(J)|}}.$$

Letting $\hat{B}_V := \frac{B_V}{B_V+2}$ we have

$$|\hat{F}_i^{N'_{M-1}+k}(J)| < \hat{B}_V \lambda_V^{N_M - N'_{M-1}+k} \frac{|V|}{|V^j|} |\hat{F}_i^{m-1}(J)|.$$

Hence we have

$$\sum_{k=1}^{N_{j+1}-N'_j} |\hat{F}_i^{N'_j+k}(J)| < C|\hat{F}_i^{m-1}(J)|.$$

We now estimate the other sums concerning intervals outside Λ as follows. Let $\mu' := \exp\left\{-\sigma'(I_0)\frac{|I_0|}{1-\rho^{-1}}\right\}$. Suppose that $F_i^{N_{M-2}}(J) \subset V^j$. Then taking the appropriate branch, $\hat{F}_i^{N_{M-2}-N'_{M-1}-1}(V) \subset V^j$ and

$$\begin{aligned} B(V, \hat{F}_i^{N_{M-2}}(J)) &< \lambda'_V B(\hat{F}_i^{N_{M-2}-N'_{M-1}-1}(V), \hat{F}_i^{N_{M-2}}(J)) \\ &< \frac{\lambda'_V}{\mu'} B(\hat{F}_i^{-1}(V), \hat{F}_i^{N'_{M-1}}(J)) \\ &< \frac{\lambda'_V}{\mu\mu'} B(V, \hat{F}_i^{N'_{M-1}+1}(J)) \end{aligned}$$

Shrinking I_0 if necessary, as usual, so that $\frac{\lambda'_V}{\mu\mu'} =: \lambda_V < 1$, we obtain

$$B(V, \hat{F}_i^{N_{M-2}}(J)) < \lambda_V B(V, \hat{F}_i^{N'_{M-1}+1}(J)).$$

Clearly then we can proceed in bounding the sum using the usual method of decaying cross–ratios. So can bound $\sum_{k=0}^{m-1} |\hat{F}_i^k(J)|$ above by $C|\hat{F}_i^{m-1}(J)|$ for this case.

To complete this case, we will bound $|\hat{F}_i^{m-1}(J)|$ in terms of $|\hat{F}_i^m(J)|$. We do this by constructing an extension. Let the left–hand and right–hand members of $F_i^{-1}(p')$ be denoted by b and b' respectively. Denote (b, b') by V'. By Lemma 5.5, V' is a $\delta_{V'}$–scaled neighbourhood of V where $\delta_{V'}$ depends only on f.

Claim. *For all domains V^j, $j > 2$ there exists an extension to some interval $U^j \supset V^j$ such that $U^j \subset V$ and $F_i^{m_{V,j}} : U^j \to V'$ is a diffeomorphism.*

Proof. For $j > 2$ the return maps are a composition of $F_i|_V$ followed by $F_i|_{I_i^R}$ and then some number of iterates of $F_i|_{I_i^L}$. So $\hat{F}_i^{-1}$ must pull V' back into I_i^L. Observe that this element of $F_i^{-1}(V')$ is below p' (and clearly away from $F_i(c)$). Any further pullbacks in I_i^L remain below p' also. Therefore when some element $F_i^{-k}(V')$ is finally pulled back into I_i^R, it is mapped above p and remains away from $F_i(c)$. Therefore we have elements of $F_i^{-k-2}(V')$ mapping inside V which don't contain c. $\square$

By the above claim and Theorem 2.4 we have some $C > 0$ depending only on f such that if $j > 2$,

$$\frac{1}{C}\frac{|V|}{|V^j|} \leqslant |D\hat{F}_i|_{V^j} \leqslant C\frac{|V|}{|V^j|}.$$

(Recall that we are assuming that $F_i^{m-1}(V) \cap \Lambda = \emptyset$.)

Therefore,

$$\sum_{k=1}^{N_M - N'_{M-1}} |\hat{F}_i^{N'_{M-1}+k}(J)| < C|\hat{F}_i^{m}(J)|.$$

There remains a further case to consider. Above we assumed $\hat{F}_i^{m-1}(J) \subset V^j$ where $j > 2$. But if $j \in \{1, 2\}$ we have two cases. We first note that if $F_i^{l_m}(J) \cap \{r, r'\} = \emptyset$ then the intervals we are concerned with are either completely inside Λ_2, Λ_1 or completely inside $V \setminus (\Lambda_2 \cup \Lambda_1)$. Then we may proceed as above. But if $F_i^k(J)$ contains r or r' then we split $F_i^k(J)$ into two intervals, with this periodic point at their intersection. We may then apply the procedure above to estimate the size of each interval. We need only apply this splitting argument once since if we intersect a periodic point of $\hat{F}_i$ once, we must stay there for all time under iteration by $\hat{F}_i$. Thus we need only alter our constants by a factor of 2 to deal with this case. Note that we only have one sum where this problem could occur: $\sum_{k=1}^{N'_M - N_M} |\hat{F}_i^{N_M+k}(J)|$ where $N'_M = m$. This is because r is a fixed point for $\hat{F}_i$.

Clearly, we can use the cross–ratio argument as usual to obtain the estimate $|F_i^{l_1}(J)| < \lambda_V^{m-1} C |F_i^{l_m}(J)|$, so $|J| < \lambda_V^{m-1} C |F_i^{l_m}(J)|$. $\square$

We may adjust our usual λ so that $\lambda_V \leqslant \lambda < 1$.

Proof of Proposition 5.3. Suppose first that $F_i^{m+1}(J) \subset I_i^j$ for $j \neq L, R$. Then, in particular, we can be sure that $F_i^m(J)$ does not contain p or p'. Then we also know that none of $F_i^k(J)$ contain p or p' for $0 \leqslant k \leqslant m - 1$. This means that we can be sure that all the intervals we consider are either contained in V or are disjoint from V.

Recall that $0 \leqslant s_1 < s_2 \leqslant s_3 = m$ are defined as follows. (We suppose that some iterate of J enters V: otherwise the proof is simpler.) $F_i^k(J) \subset I_i' \setminus V$ for $1 \leqslant k \leqslant s_1$; $F_i^{s_1+1}(J) \subset V \cap (I_i^L \cup I_i^R)$; and $F_i^{s_2}(J) \subset V \cap (I_i^L \cup I_i^R)$, $F_i^{s_2+k}(J) \subset I_i' \setminus V$ for $1 \leqslant k \leqslant s_3 - s_2$.

Then if $s_3 > s_2$,

$$\sum_{k=1}^{s_3-s_2} |F_i^{s_2+k}(J)| < |F_i^{s_3}(J)| \sum_{k=0}^{s_3-s_2-1} \gamma^{-k} < C|F_i^{s_3}(J)|,$$

by Lemma 5.4.

From Lemma 5.6,

$$\sum_{k=1}^{s_2-s_1} |F_i^{s_1+k}(J)| < C|F_i^{s_2}(J)|$$

and $|F_i^{s_1+1}(J)| < C|F_i^{s_2}(J)|$.

Also

$$\sum_{k=0}^{s_1} |F_i^k(J)| \;\; < \;\; \gamma^{-1}|F_i^{s_1+1}(J)| \sum_{k=0}^{s_1} \gamma^{-k} < C|F_i^{s_2}(J)|.$$

Therefore,

$$\sum_{k=0}^{s_2} |F_i^k(J)| < C|F_i^{s_2}(J)|.$$

If $s_3 > s_2$ then

$$\sum_{k=0}^{s_3} |F_i^k(J)| < C\left(|F_i^{s_3}(J)| + |F_i^{s_2}(J)|\right).$$

Therefore, the first part of the proposition is proved.

Now if $F_i^{m+1}(J) \subset I_i^j$ for $j \neq L, R, 0$ then recalling that $s_3 = m$ we will obtain an estimate for $|F_i^{s_2}(J)|$ in terms of $B(I_i, F_i^{m+1}(J))$.

$$B(I_i, F_i^{s_2}(J)) < B(F_i^{-s_3+s_2}(I_i), F_i^{s_2}(J)) < \frac{B(I_i, F_i^m(J))}{\mu} < \frac{B(I_i, F_i^{m+1}(J))}{\mu^2}.$$

We are allowed to use μ here since all intermediate intervals must be disjoint (otherwise we would have to pass through V again). Therefore

$$|F_i^{s_2}(J)| < \frac{|I_i|}{1 + \frac{2\mu^2}{B(I_i, F_i^{m+1}(J))}} < C|I_i|B(I_i, F_i^{m+1}(J)).$$

Similarly we can show that $|F_i^m(J)| < C|I_i|B(I_i, F_i^{m+1}(J))$. Therefore

$$\sum_{k=0}^{s_3} |F_i^k(J)| < C|I_i|B(I_i, F_i^{m+1}(J)) < C_1|I_i|$$

for some $C_1 > 0$.

We now prove the final part of the proposition. Clearly for any run of intervals $F_i(J), \ldots, F_i^k(J) \subset I_i^L \cup I_i^R$, considering the branch of F_i^{-k} which follows the iterates of J, we have $B(F_i^k, F_i^{-k}(I_i), J) > \mu''$ where $\mu'' := \exp\{-C_1\sigma'(|I_0|)|I_0|\}$. We consider the branch of F_i^{-m-2} which follows the backward orbit of $F_i^{m+1}(J)$. Clearly, $F_i^{-m-2}(I_i)$ is strictly inside I_i^j. Thus,

$$\begin{aligned} B(I_i, J') &< \lambda' B(I_i^j, J') < \lambda' B(F_i^{-m-2}(I_i), J') < \frac{\lambda'}{\mu''} B(F_i^{-1}(I_i), F_i^{m+1}(J')) \\ &< \frac{\lambda'}{\mu''\mu} B(I_i, F_i^{m+2}(J')). \end{aligned}$$

For $|I_0|$ small enough, we can alter the usual λ slightly so that $\frac{\lambda'}{\mu''\mu} \leqslant \lambda$ and still ensure that $\lambda < 1$. Thus, $B(I_i, J') < \lambda B(I_i, F_i^{m+2}(J'))$ as required.

When we do not escape $I_i^L \cup I_i^R$ then we may have some intersection with p or p'. In this case, we split our interval in two and estimate the size of each piece as above. We need only apply this idea once, so we can change our constants to cater for this case too. In this case, part 2 of the proposition doesn't occur. $\square$

6. Proof of the main theorem in the non–infinitely renormalisable case

We recall that $B(f^n, T, J) > \exp\{-C\sum_{k=0}^{n-1}|f^k(T)|^{1+\eta}\}$ when $f \in C^{2+\eta}$. We will find a bound on the sum $\sum_{k=0}^{n-1}|f^k(T)|^{1+\eta}$ by using the main propositions above and also finding some decay property for the size of the domains of F_i for some values of i. We assume that $f^k(T) \cap \partial I_j \neq \emptyset$ only within a cascade case (i.e. when there exist i, m such that F_i is in a cascade case and $f^k(T) \subset I_i \setminus I_{i+m}$). It is easy to see how to extend the proof when this is not true.

Let $F_i : \bigcup_j U_i^j \to I_i$ be the first entry map to I_i (we include the branches of the first return map in this case too). For $i < j$ and an interval V, we define $S(i, j, V)$ to be the maximum of $|f^{i+1}(V)|, |f^{i+2}(V)|, \ldots, |f^j(V)|$. We will consider $S(n_{i+1}, n_i, T)$. Let $n(i, j)$ be such that $F_i|_{U_i^j} = f^{n(i,j)}|_{U_i^j}$. Now let $U_i^{s(i)}$ be the interval for which $S\left(0, n(i, j), U_i^j\right)$ is maximal. Let $\hat{n}(i) = n(i, s(i))$. Clearly,

$$S(n_{i+1}, n_i, T) \leqslant S\left(0, \hat{n}(i), U_i^{s(i)}\right).$$

We would like to show that for certain i, this quantity decays with i in a controlled way.

We start by assuming that F_{i-1} is in a well bounded case. We have two cases. Firstly, suppose that $U_i^{s(i)} \subset I_i$. Then since F_{i-1} is in a well bounded

case, we have $|U_i^{s(i)}| < \frac{|I_{i-1}|}{1+2\chi}$. Since I_i is a domain of the first return map to I_{i-1} we have

$$|U_i^{s(i)}| < \frac{S\left(0, \hat{n}(i-1), U_{i-1}^{s(i-1)}\right)}{1+2\chi}.$$

Now assume that $U_i^{s(i)} \cap I_i = \emptyset$. Then there exists some extension $V_i \supset U_i^{s(i)}$ such that $f^{n(s(i))} : V_i \to I_{i-1}$ is a diffeomorphism. We will show that $U_i^{s(i)}$ is uniformly smaller than V_i. By (1) we know that $B(V_i, U_i^{s(i)}) < \frac{B(I_{i-1}, I_i)}{\mu}$ for μ as in (4). Thus, by Lemma 3.1, $|U_i^{s(i)}| < \frac{|V_i|}{1+\frac{2\mu}{\Delta(\chi)}}$. Since V_i is a first return domain to I_{i-1} we have

$$|U_i^{s(i)}| < \frac{S\left(0, \hat{n}(i-1), U_{i-1}^{s(i-1)}\right)}{1+\frac{2\mu}{\Delta(\chi)}}.$$

Let $\gamma := \max\left(\frac{1}{1+2\chi}, \frac{1}{1+\frac{2\mu}{\Delta(\chi)}}\right)$. Clearly $\gamma < 1$. So

$$S\left(0, \hat{n}(i), U_i^{s(i)}\right) < \gamma S\left(0, \hat{n}(i-1), U_{i-1}^{s(i-1)}\right).$$

We let $C_{all} = \max(C_{wb}, C_{casc}, 3C_{ex})$. Note that by disjointness, all $\sigma_i, \sigma_{i,m} < 1$. If $f \in NF^{2+\eta}$ and F_{i-1} is well bounded, we have

$$\begin{aligned} & B(f^{n_i - n_{i+1}}, f^{n_{i+1}+1}(T), f^{n_{i+1}+1}(J)) \\ & \qquad \geqslant \exp\left\{-C\left(S(n_{i+1}, n_i, T)\right)^\eta \sum_{k=1}^{n_i - n_{i+1}} |f^{k+n_{i+1}}(T)|\right\} \\ & \qquad > \exp\left\{-C\left(S\left(0, \hat{n}(i), U_i^{s(i)}\right)\right)^\eta C_{all}\right\} \\ & \qquad > \exp\left\{-C\left(\gamma S\left(0, \hat{n}(i-1), U_{i-1}^{s(i-1)}\right)\right)^\eta C_{all}\right\}. \end{aligned}$$

If we are not in the infinite cascade case then the sums for $F_i, F_{i+1}, \dots$ can be broken into blocks consisting of a cascade; possibly followed by an exceptional case; followed by one or more well bounded cases. So suppose that F_i is well bounded, $F_i, F_{i+1}, \dots, F_{i+m-1}$ have central returns, F_{i+m} has a non–central return and F_{i+m+1} is an exceptional case. So note that, in particular, F_{i+m+2} must be well bounded. Then,

$$\begin{aligned} & S\left(0, \hat{n}(i+m+3), U_{i+m+3}^{s(i+m+3)}\right) < \gamma S\left(0, \hat{n}(i+m+2), U_{i+m+2}^{s(i+m+2)}\right), \dots \\ & \qquad \dots, \gamma S\left(0, \hat{n}(i+1), U_{i+1}^{s(i+1)}\right) < \gamma^2 S\left(0, \hat{n}(i), U_i^{s(i)}\right). \end{aligned}$$

Therefore, we have

$$\begin{aligned} B(f^n, T, J) &> \exp\left\{-C\sum_{k=0}^{n-1}|f^k(T)|^{1+\eta}\right\} \\ &> \exp\left\{-CC_{all}\left(S\left(0, \hat{n}(0), U_0^{s(0)}\right)\right)^\eta \sum_{k=0}^{\infty}\gamma^{k\eta}\right\} \\ &> \exp\left\{-CC_{all}\frac{(\sigma'(|I_0|))^\eta}{1-\gamma^\eta}\right\}. \end{aligned}$$

Hence it is easy to see that for any $0 < K < 1$, if I_0 is the central domain of a first return map to some I_{-1}, I_0 is sufficiently small and F_{-1} is non–central, then we may bound $B(f^n, T, J)$ below by K.

Note that we can always start with a well–bounded case when we don't have an infinite cascade. We simply induce on a nice interval finitely many times until we obtain a non–central return and thus obtain a suitable I_{-1}. We consider the infinite cascade case in the next section.

The second part of Theorem 1.2, concerning $A(f^n, T, J)$, is proved in the same way.

7. Infinite cascade case

Here we consider the case where we have some I_0 such that F_i are central for $i = 0, 1, \ldots$. In this case we will find that $\frac{|I_{i+1}|}{|I_i|}$ gets very close to 1. See Figure 5 for an example of such a map. In particular, I_i will not shrink down to a point (the critical point c) as i increases so we can't use the method above to bound sums of intervals which land very close to c. The principal tool here is an extension given by a result of [**K2**]. We will not supply all the details of our proof of Theorem 1.2 in this case since the techniques are mostly the same as applied in the previous sections. We start by letting I_0 be any nice interval about c. We assume that we have some infinite cascade. This means that for a nice interval $I_0 \ni c$, F_i is central (and high) for all i, where F_i is defined in the usual way. The main idea here is that we can still find good bounds on some interval $I_{0,0}$ and then apply the methods of Section 4 to it. Then we need to find another interval $I_{1,0}$ around c which is smaller than all $I_{0,i}$, also has good bounds and is uniformly smaller than $I_{0,0}$. In such a way, we obtain a sequence of intervals $I_{i,0}$ which can each be treated as in the high cascade case above, and which shrink uniformly to the critical point. Clearly $F_{i,j}$ will always be central and high for all $i, j \geqslant 0$.

Proposition 7.1. *For $f \in NF^2$, and $\xi > 0$ there exists some $C_{inf} > 0$ such that for any small $I_{0,0}$ defined as above, $T \subset I_{0,0}$ implies*

$$\sum_{k=0}^{n-1}|f^k(T)|^{1+\xi} < C_{inf}.$$

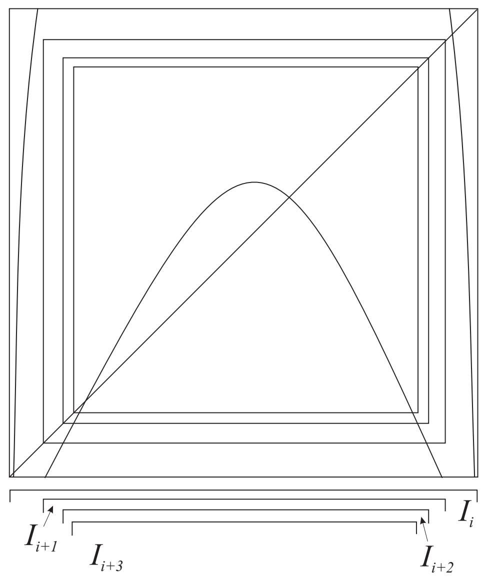

FIGURE 5. An infinite cascade.

Clearly this completes the proof of Theorem 1.2 in this case.

Proof. We will prove this with a series of lemmas.

For all i the central branch of F_i has two fixed points, q_0 and p_0 to the left and right of c respectively (as usual, we assume that $F_i(c)$ is a maximum for $F_i|_{I_{i+1}}$). We let q_0' be the point in I_{i+1} not equal to q_0 which maps by F_i to q_0. We define p_0' similarly. We define $I_{0,0}$ to be (p_0', p_0). Let $F_{0,0} : \bigcup_j I_{0,0}^j \to I_{0,0}$ be the first return map to $I_{0,0}$ (where $I_{0,0}^0$ is the central domain). We have the following lemma.

Lemma 7.2. *There exists some $\hat{\chi} > 0$ depending only on f such that $I_{0,0}$ is a $\hat{\chi}$–scaled neighbourhood of every domain $I_{0,0}^j$ which has $\partial I_{0,0}^j \cap \partial I_{0,0} = \emptyset$.*

Proof. Clearly, I_i tends to (q_0, q_0'). So we denote (q_0, q_0') by I_∞. We will first show that I_∞ is uniformly larger than $I_{0,0}$ and then show that all except two non–central domains of the first entry map to $I_{0,0}$ have an extension to I_∞ and show what this means for $I_{0,0}^j$. These two domains are the ones with either p_0 or p_0' in the closure.

In a similar manner to the exceptional case, we will find an upper bound for $|DF_i|_{I_{i+1}}$. This will allow us to get good bounds for the first return map to $I_{0,0}$

For large i, the ratio I_i has $\frac{|I_{i+1}|}{|I_i|}$ close to 1. The following lemma, an adaptation of Lemma 7.2 of [**K2**], allows us to bound $|DF_i|_{I_{i+1}}$.

Lemma 7.3. *If $f \in NF^2$ then there exist constants $0 < \tau_2 < 1$ and $\tau_3 > 0$ with the following property. If T is any sufficiently small nice interval around the critical point, R_T is the first entry map to T and its central domain J is sufficiently big, i.e. $\frac{|J|}{|T|} > \tau_2$, then there is an interval W which is a τ_3–scaled neighbourhood of the interval T such that if $c \in R_T(J)$ then the range of any branch of $R_T : V \to T$ can be extended to W provided that V is not J.*

This lemma is only needed as a C^3 result in [**K2**], but it easily extends to our C^2 case.

It is straightforward to see that the above lemma is sufficient to prove a version of Lemma 5.5 in our case. That is, for large i, there exists some $\hat{C}'$ such that $|DF_i|_{I_{i+1}} < \hat{C}'$. This implies that there exists some $0 < \theta < 1$ depending only on f such that $|I_{0,0}| < \theta|I_\infty|$ and, equivalently, some $\delta > 0$ such that I_∞ is a δ–scaled neighbourhood of $I_{0,0}$.

Now, for the moment we let $F_{0,0}$ also denote the first entry map and $\bigcup_j I^j_{0,0}$ also include the first entry domains. We will show that many of the branches have an extension to a uniformly larger domain. Suppose that there exists a domain $I^j_{0,0}$ with $\overline{I_{0,0}} \cap \overline{I^j_{0,0}} = \emptyset$ such that $F_{0,0} : I^j_{0,0} \to I_{0,0}$ does not have an extension to I_∞. That is, supposing $F_{0,0}|_{I^j_{0,0}} = f^{n(j)}|_{I^j_{0,0}}$, there is no interval $V \supset I^j_{0,0}$ such that $f^{n(j)} : V \to I_\infty$ is a diffeomorphism. Let $0 \leqslant k \leqslant n(j) - 1$ be maximal such that $f^{n(j)-k} : f^k(I^j_{0,0}) \to I_{0,0}$ has no extension to I_∞. Clearly if $I_{0,0}$ is small $f : f^{n(j)-1}(I^j_{0,0}) \to I_{0,0}$ always has an extension, so $k < n(j) - 1$. Then there exists some interval $W \supset f^{k+1}(I^j_{0,0})$ such that $f^{n(j)-k-1} : W \to I_\infty$ is a diffeomorphism and the element W' of $f^{-1}(V)$ containing $f^k(I^j_{0,0})$ contains c.

Since I_∞ is a nice interval, $W' \subset I_\infty$. We also know that $f^k(I^j_{0,0}) \subset I_\infty \setminus I_{0,0}$. Therefore W' contains either p_0 or p_0'. But then either $f^{n(j)-k-1}(p_0)$ or $f^{n(j)-k-1}(p_0')$ is contained in $I_\infty \setminus I_{0,0}$ which is not possible.

Consider $I^j_{0,0}$ for some $j \neq 0$ where $I^j_{0,0} \subset I_{0,0}$ is a domain of the first return map. We will show that this domain is uniformly deep inside $I_{0,0}$. There exists some $V \supset f(I^j_{0,0})$, where $f^{n(j)} : V \to I_\infty$ is a diffeomorphism and V is a $\tilde{\delta}$–scaled neighbourhood of $f(I^j_{0,0})$. Let V' be the maximal interval around $I^j_{0,0}$ such that $f(V') = V$. We show that $V' \subset I_{0,0}$. Let $V(f(c))$ denote the maximal interval around $f(c)$ which pulls back by f^{-1} to $I_{0,0}$. If V is not contained in $V(f(c))$ then either p_0 or p_0' is contained in V'. Thus, $f^{n(j)}(p_0)$ or $f^{n(j)}(p_0')$ lies in $I_\infty \setminus I_{0,0}$, a contradiction. So $V' \subset I_{0,0}$ and $I_{0,0}$

is a δ'–scaled neighbourhood of $I_{0,0}^j$ where $\delta' = \min\left(\tilde{\delta}, \frac{1}{2}\right)$. The case of the central branch follows in the usual manner. $\square$

So we are in a type of high cascade case for $F_{0,0}$. Note that the branches with p_0 or p_0' in their closure can be dealt with in the same way as the domains V^1, V^2 were dealt with in the exceptional case.

We may assume that $F_{0,0}$ has an infinite cascade and is high too. Let $F_{0,1}$ be the first return map to $I_{0,0}$ and so on, so we obtain $I_{0,i}$. We sum for $F_{0,0}, F_{0,1}, \ldots$ as in the high cascade case. We let q_1, q_1', p_1, p_1' be defined as above for the fixed points of $F_{0,0}|_{I_{0,1}}$. We let $I_{0,\infty}$ denote (q_1, q_1'). We may apply the same ideas as above to find some new interval $I_{1,0} := (p_1, p_1')$ which has $|I_{1,0}| < \theta |I_{0,\infty}|$. We may define $I_{i,j}$ for $i \geqslant 2$, and $0 \leqslant j \leqslant \infty$ in a similar way.

Let $f^{N_i}(T)$ be the last iterate of T which lies inside $I_{i,0}$. Let N_i' be the maximal integer $N_i \geqslant N_i' > N_{i+1}$ such that $f^{N_i'}(T)$ is not in $I_{i,0} \setminus I_{i,\infty}$. Then these arguments prove the following lemma.

Lemma 7.4. *There exists some $C > 0$ such that*

$$\sum_{k=1}^{N_i - N_i'} |f^{k+N_i'}(T)| < C\hat{\sigma}_i$$

where $\hat{\sigma}_i$ is defined as follows. Let $\sigma_i := \sup_{V \in \mathrm{dom} F_{i,0}} \sum_{j=1}^{n(V)} |f^j(V)|$ (and $n(V)$ is defined as k where $F_{i,0}|_V = f^k$). Let $\hat{V} \subset I_{i,0} \setminus I_{i,1}$ be an interval such that $f^{\hat{n}}(\hat{V})$ is one of the connected components of $I_{i,0} \setminus I_{i,1}$ and $f^j(\hat{V})$ is disjoint from both $I_{i,0} \setminus I_{i,1}$ and $I_{i+1,0}$ for $0 < j < \hat{n}(\hat{V})$. Then $\hat{\sigma}_i$ is the supremum of all such sums $\sum_{j=1}^{\hat{n}(\hat{V})} |f^j(\hat{V})|$ and σ_i.

Now we consider $\sum_{k=1}^{N_i' - N_{i+1}} |f^{k+N_{i+1}}(T)|$. If none of these intervals contain p_i, q_i then we are in $I_{i,\infty} \setminus I_{i+1,0}$. By the Minimum principle, $|DF_i|_{I_{i,\infty} \setminus I_{i+1,0}}$ is uniformly greater than 1. So we can easily bound our sum. If none of our intervals contains p_i, but some $f^{k+N_{i+1}}(T)$ contains q_0, q_0' we can split $f^{k+N_{i+1}}(T)$ at q_0 or q_0' into two intervals. It is easy to see that there is some $C > 0$ such that $\sum_{k=1}^{N_i' - N_{i+1}} |f^{k+N_{i+1}}(T)| < C\hat{\sigma}_i$. If p_0, p_0' is contained in some $|f^{k+N_{i+1}}(T)|$ then we must split the interval at p_0 or p_0'. Note that we may have to split the interval $|f^{k+N_{i+1}}(T)|$ at arbitrarily many p_i, q_i' or p_i', q_i. Therefore, $\sum_{k=1}^{N_i' - N_{i+1}} |f^{k+N_{i+1}}(T)|^{1+\xi} < C \sum_{k=i}^{\infty} (k - i) S(N_k, \hat{n}(k), T)^{\xi} \hat{\sigma}_k$ where S and $\hat{n}$ are defined analogously to Section 6. As before there is some constant $0 < \theta' < 1$, here depending on θ rather than γ such that θ' governs the decay of $S(N_i, \hat{n}(i), T)$. Hence, we can put this

estimate together with Lemma 7.4 to get

$$\sum_{k=1}^{N_i-N_{i+1}} |f^{k+N_{i+1}}(T)|^{1+\xi} < CS(N_i,\hat{n}(i),T)^{\xi}\sum_{k=0}^{\infty} k\theta'^{k\xi}.$$

Similarly to before, we can conclude that there exists some $C_{inf} > 0$ such that

$$\sum_{k=0}^{n} |f^k(T)|^{1+\xi} < C_{inf}.$$

□

Appendix A. Proof of the Yoccoz Lemma

We recall the lemma.

Lemma 4.3 *Suppose that $f \in NF^2$. Then for all $\delta, \delta' > 0$ there exists $C > 0$ such that if I_0 is a nice interval such that*

1. *I_0 is a δ–scaled neighbourhood of I_1;*
2. *F_i is low and central for $i = 0, \dots, m$;*
3. *there is some $0 < i < m$ with $\frac{|I_i|}{|I_{i+1}|} < 1 + \delta'$,*

then for $1 \leqslant k < m$,

$$\frac{1}{C}\frac{1}{\min(k,m-k)^2} < \frac{|I_{i+k-1} \setminus I_{i+k}|}{|I_i|} < \frac{C}{\min(k,m-k)^2}.$$

For similar statements see [**FM**] and [**Sh2**].

Proof. We first point out the following claim.

Claim 1. *For f as in the lemma, there exists some $C(f,\delta,\delta') > 0$ such that*

$$\frac{|I_m|}{|I_0|} > C(f,\delta,\delta').$$

This is proved in Section 5 of [**Sh2**]. One consequence of this is that $\frac{|I_m \setminus I_{m+1}|}{|I_0|}$ is uniformly bounded below. This is one of the assumptions in the statement of the Yoccoz Lemma in [**FM**].

Our proof now involves using a result of [**ST**], the bound δ and the small size of I_0, to find a nearby map in the Epstein class. The structure of such maps, particularly at parabolic fixed points, along with some new coordinates, give us estimates for $\frac{|I_{i+k-1} \setminus I_{i+k}|}{|I_i|}$.

We suppose that $s > 0$ is such that $F_0|_{I_1} = f^s|_{I_1}$. We observe that f^{s-1} has uniformly bounded distortion depending on δ. We will denote $F_0|_{I_1}$ by

F. Letting $\psi : [a_m, a_1] \to [0,1]$ be an affine diffeomorphism we will work with the map $\psi \circ F \circ \psi^{-1}$. For the rest of the appendix we will abuse the notation and denote this map by F too.

Previously we assumed that $F|_{I_1}$ had a maximum at c. It will be convenient to suppose now for this section that c is a minimum for $F|_{I_1}$. Also we let $I_i = (a_i', a_i)$. So in particular, $F(a_{i+1}) = a_i$. We firstly define a point which allows us to partition $[a_m, a_1]$ in another way.

Let $x_0 \in [a_m, a_1]$ be defined so that $|F(x_0) - x_0| = \min_{a_m \leqslant x \leqslant a_0} |F(x) - x|$. It is easy to show that $DF(x_0) = 1$. We will suppose throughout that $|F(x_0) - x_0|$ shrinks to zero as $|I_0| \to 0$: otherwise the proof is much simpler. We can estimate the shape of F near x_0 using the following definition and lemma.

Let $\kappa > 0$. We say that the real analytic map $f : [0,1] \to [0,1]$ is in the *Epstein class* $\mathcal{E}_\kappa$ if $f(x) = \varphi Q \psi$ where Q is the quadratic map $Q(z) = z^2$, ψ is an affine map and $\varphi : [0,1] \to [0,1]$ is a diffeomorphism whose inverse has a holomorphic extension which is univalent in the domain $\mathbb{C}_{(-\kappa, 1+\kappa)} := \mathbb{C} \setminus ((-\infty, -\kappa] \cup [1+\kappa, \infty))$. For more details on maps in this class see [**MS**]. The following lemma is proved in [**ST**].

Lemma A.1. *Let $f \in NF^2$. Suppose that I is a nice interval around c and J is a first entry domain which is disjoint from I and with entry time s. Suppose that $\delta > 0$ is some constant such that there exists some $\hat{J} \supset J$ such that $f^s : \hat{J} \to I'$ is a diffeomorphism where I' is a κ–scaled neighbourhood of I and $\sum |f^j(\hat{J})| \leqslant 1$. Let $\tau_0 : J \to [0,1]$ and $\tau_s : I \to [0,1]$ be affine diffeomorphisms. Then for all $\epsilon > 0$ there exists $\delta > 0$ such that $|I| < \delta$ implies that there exists some function $G : I \to I$ in the Epstein class $\mathcal{E}_{\frac{\kappa}{2}}$ such that $\|\tau_s \circ f^s \circ \tau_0^{-1} - G\|_{C^2} < \epsilon$.*

We use this to prove the following claim.

Claim 2. *There exists some $0 < A < B$ such that for I_0 sufficiently small*

$$F(x_0) + (x - x_0) + A(x - x_0)^2 \leqslant F(x) \leqslant F(x_0) + (x - x_0) + B(x - x_0)^2.$$

Proof. We know that $f^s : I_2 \to I_1$ has the following property. The map $f^{s-1} : f(I_2) \to I_1$ has an extension to I_0. Furthermore, since I_0 is a δ–scaled neighbourhood of I_1 we use Lemma A.1 to obtain a map G_∞ in the Epstein class which is C^2–close to f^s.

In fact we can choose different starting intervals I_n with the same real bounds which are smaller and smaller and which are then rescaled to maps F_n which map from the unit interval to itself. For each such map we obtain the nearby map G_n in the Epstein class where $\|F_n - G_n\|_{C^2} \to 0$ as $n \to \infty$.

For F_n we let x_0^n denote a point which is equivalent to x_0 for F. Since we assume that $|F_n(x_0^n) - x_0^n|$ goes to zero, our limit map G_∞ has a parabolic fixed point x_0^∞. Also $D^2G_\infty(x_0^\infty) > 0$. Thus, there exist $0 < A < B$ depending only on f such that for all $x \in [0,1]$ we have

$$G_\infty(x_0^\infty)+(x-x_0^\infty)+A(x-x_0^\infty)^2 \leqslant G_\infty(x) \leqslant G_\infty(x_0^\infty)+(x-x_0^\infty)+B(x-x_0^\infty)^2.$$

Clearly, for large n, we have the same condition for G_n. Therefore, if we take I_0 small enough, we may assume that it holds for F too. $\square$

We denote $\epsilon := F(x_0) - x_0$. Then we have

$$\epsilon + A(x-x_0)^2 \leqslant F(x) - x \leqslant \epsilon + B(x-x_0)^2.$$

We suppose that N is such that $x_0 \in [a_N, a_{N+1})$. Then for $0 \leqslant i \leqslant N-1$ we let $x_i := F^i(x_0)$. We will use this equation to find estimates for $a_j - a_{j+1}$. Throughout we will let C, C' denote some constants depending only on δ, δ'.

Claim 3.

$$N = O\left(\frac{1}{\sqrt{\epsilon}}\right).$$

Proof. Let $N' = \max\{1 \leqslant j \leqslant N-1 : x_j - x_0 \leqslant \sqrt{\epsilon}\}$. We will first show that N' satisfies the claim. For $j \leqslant N'$, we have

$$\epsilon \leqslant x_{j+1} - x_0 \leqslant \epsilon(B+1).$$

Therefore,

$$N'\epsilon \leqslant \sum_{j=0}^{N'-1} x_{j+1} - x_j \leqslant N'\epsilon(B+1).$$

Since $\sum_{j=0}^{N'-1} x_{j+1} - x_j = x_{N'} - x_0 \leqslant \sqrt{\epsilon}$ we have $N' \leqslant \frac{1}{\sqrt{\epsilon}}$. Furthermore, $x_{j+1} - x_0 > \sqrt{\epsilon}$ so $\epsilon(N'(B+1)+1) > \sqrt{\epsilon}$ and $N' > \frac{1}{(B+1)\sqrt{\epsilon}} - 1$. I.e. $N' = O\left(\frac{1}{\sqrt{\epsilon}}\right)$.

Next we find estimates for $N - N'$. For $N' < j \leqslant N$ we again consider the equation

$$\epsilon + A(x_j - x_0)^2 \leqslant x_{j+1} - x_j \leqslant \epsilon + B(x_j - x_0)^2.$$

But note that here $B(x_j - x_0)^2 > \epsilon$ so we can write instead

$$A(x_j - x_0)^2 \leqslant x_{j+1} - x_j \leqslant 2B(x_j - x_0)^2.$$

We make a change of coordinates. We let $y_j := \frac{1}{x_j - x_0}$. Then we have

$$y_j - y_{j+1} = \frac{x_{j+1} - x_j}{(x_j - x_0)(x_{j+1} - x_0)}.$$

By the above bounds we have

$$\frac{A(x_j - x_0)}{x_{j+1} - x_0} < y_j - y_{j+1} < \frac{2B(x_j - x_0)}{x_{j+1} - x_0} < 2B.$$

Furthermore,

$$y_j - y_{j+1} > \frac{A(x_j - x_0)}{(x_{j+1} - x_j) + (x_j - x_0)} > \frac{A(x_j - x_0)}{2B(x_j - x_0)^2 + (x_j - x_0)} > \frac{A}{2B+1}.$$

Observe that $x_N \in (a_1, a_0)$ and $|a_0 - a_1| > \delta$. So since $|x_N - x_{N-1}|$ is approximately $|a_0 - a_1|$ and since we fixed δ, we know that $y_N = O(1)$. Also note that $y_{N'} = O\left(\frac{1}{\sqrt{\epsilon}}\right)$ and so $y_{N'} - y_N = O\left(\frac{1}{\sqrt{\epsilon}}\right)$. Summing we obtain

$$\frac{C}{\sqrt{\epsilon}} < y_{N'} - y_N = \sum_{j=N-1}^{N'} y_j - y_{j+1} < 2B(N - N')$$

and

$$\frac{C'}{\sqrt{\epsilon}} > y_{N'} - y_N = \sum_{j=N-1}^{N'} y_j - y_{j+1} > \frac{A(N - N')}{2B+1}.$$

So $N - N' = \left(\frac{1}{\sqrt{\epsilon}}\right)$ too. Adding this to the estimates for N' we prove the claim. $\square$

To prove the lemma, we will use Claims 1 and 3 together, along with bounded distortion, which means that $a_j - a_{j+1}$ is like $x_{N-j} - x_{N-j-1}$.

Firstly we will use the above coordinate change again. For $j > N'$ we have

$$y_j > y_j - y_N = \sum_{j=N-1}^{j} y_i - y_{i+1} > \frac{A(N-j)}{2B+1}$$

and so $\frac{1}{x_j - x_0} > \frac{A(N-j)}{2B+1}$ and $x_{j+1} - x_j < 2B\left(\frac{2B+1}{A(N-j)}\right)^2$.

We have proved that if $0 \leqslant j \leqslant N'$ then

$$\epsilon < x_{j+1} - x_j < C'\epsilon \tag{5}$$

and if $N' < j \leqslant N$ then

$$\epsilon < x_{j+1} - x_j < \frac{C'}{(N-j)^2}. \tag{6}$$

Similarly we can define $x_j = F^j(x_0)$ for negative j where $0 \leqslant |j| < m - N$. Now we will show that Claim 3 follows for this situation too and we get equivalents to (5) and (6). We define some M' analogously to the definition for N' and so if $|j| \leqslant M'$ then

$$\epsilon < x_{j+1} - x_j < C'\epsilon.$$

And if $M' < |j| \leqslant m - N$ then

$$\frac{C}{(m-N+j)^2} < x_{j+1} - x_j < \frac{C'}{(m-N+j)^2}.$$

(In the step of the proof where estimates on y_{N-m} are required, we use Claim 1 to give $|a_{m-1} - a_m|$ uniformly bounded below and the fact that

$|x_{-m-1} - x_m|$ is approximately $|a_{m-1} - a_m|$.) Note also that we can show that $m - M' = O\left(\frac{1}{\sqrt{\epsilon}}\right)$.

Observe that $a_j - a_{j+1}$ is essentially the same as $x_{N-j} - x_{N-j-1}$. So if $N \geqslant j \geqslant N - N'$, we have

$$C\epsilon < a_j - a_{j+1} < C'\epsilon.$$

Observe that $\frac{1}{N-N'} \geqslant \frac{1}{j} \geqslant \frac{1}{N}$. Since $\epsilon\left(\frac{1}{N^2}\right)$ and $\epsilon\left(\frac{1}{(N-N')^2}\right)$ this implies that we have

$$\frac{C}{j^2} < a_j - a_{j+1} < \frac{C'}{j^2}.$$

Now if $N - N' \geqslant j \geqslant O(1)$ then clearly we have $a_j - a_{j+1} < \frac{C'}{j^2}$. Also,

$$x_{N-j} - x_{N-j-1} > A(x_{N-j-1} - x_0)^2 = A\left(\sum_{k=j-1}^{N-1} x_{N-k} - x_{N-k-1}\right)^2$$
$$\geqslant A\left(\sum_{k=1}^{N'} x_k - x_{k-1}\right)^2 \geqslant A(N'\sqrt{\epsilon})^2.$$

Now since $\sqrt{\epsilon} = O\left(\frac{1}{N'}\right)$, we have $x_{N-j} - x_{N-j-1} \gtrsim 1$. Thus

$$\frac{C}{j^2} < a_j - a_{j+1} < \frac{C'}{j^2}.$$

If $N \leqslant j \leqslant m - M'$ then again we have

$$C\epsilon < a_j - a_{j+1} < C'\epsilon.$$

Note that we also have $m - N \geqslant m - j \geqslant m - M'$. Since $m - N, m - M' = \left(\frac{1}{\sqrt{\epsilon}}\right)$ we have

$$\frac{C}{(m-j)^2} < a_j - a_{j+1} < \frac{C'}{(m-j)^2}.$$

If $m - M' \leqslant j \leqslant m - 1$ we have

$$\frac{C}{(m-j)^2} < a_j - a_{j+1} < \frac{C'}{(m-j)^2}$$

where the lower bound follows as above.

To conclude, if $1 \leqslant j \leqslant N$ then we have some constant C such that $j \leqslant C(m-j)$ and $a_j - a_{j+1} \asymp \frac{1}{j^2}$. If $N \leqslant j \leqslant m-1$ then we have some constant C' such that $m - j \leqslant C'j$ and $a_j - a_{j+1} \asymp \frac{1}{(m-j)^2}$. So in either case we have

$$a_j - a_{j+1} \asymp \frac{1}{(\min(j, m-j))^2}$$

as required. $\square$

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